

A COVARIANCE IDENTITY FOR MARGINAL RÉNYI ENTROPY OF PLANAR CONVEX BODIES

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ABSTRACT. For a uniform point $\mathbf{X}$ in a convex polygon and every $0 < \alpha < \infty$, we prove the covariance identity

$$\frac{r_\alpha + r''_\alpha}{r_\alpha} = -\alpha \operatorname{Cov}(g', k')$$

away from directions normal to edges. Here $r_\alpha(\theta) = \exp h_\alpha(\langle \mathbf{u}_\theta, \mathbf{X} \rangle)$, where $\mathbf{u}_\theta = (\cos \theta, \sin \theta)$ and h_α is Rényi entropy; derivatives of r_α are angular, while g' and k' are respectively the upper and lower boundary slopes when the projection direction is taken as the horizontal axis. The covariance is taken under the density proportional to the α -th power of the chord length and is nonpositive because the slopes are oppositely monotone. Together with an explicit derivative-jump formula at edge normals, the identity gives a new proof that, for $\mathbf{X}$ uniform on any planar convex body, the functional $\mathbf{v} \mapsto \exp h_\alpha(\langle \mathbf{v}, \mathbf{X} \rangle)$, extended by 0 at the origin, is a norm for every $0 \leq \alpha \leq \infty$. A corresponding one-sided measure inequality for arbitrary planar convex bodies yields a geometric characterization: at each finite positive order, the unit ball under this norm is a polygon if and only if the original body is a triangle, in which case the unit ball is a centrally symmetric hexagon.

1. INTRODUCTION

For a real random variable W with density f , its *Rényi entropy* is

$$h_\alpha(W) = \frac{1}{1-\alpha} \log \int_{\mathbb{R}} f^\alpha \quad (\alpha \neq 1), \quad h_1(W) = - \int_{\mathbb{R}} f \log f,$$

where $0 \log 0 = 0$, $0 < \alpha < \infty$, and all logarithms are natural. The order 1 is Shannon differential entropy. At the endpoint orders, $h_0(W) = \log |\{f > 0\}|$ and $h_\infty(W) = -\log \|f\|_\infty$.

Let $\mathbf{X}$ be uniform on a planar *convex body* K , i.e., a compact convex set with nonempty interior. We write $|K|$ for K 's area and use sans serif Latin letters for vectors, with capitals for random vectors. For $0 \leq \alpha \leq \infty$, set

$$N_\alpha(\mathbf{v}) = e^{h_\alpha(\langle \mathbf{v}, \mathbf{X} \rangle)} \quad (\mathbf{v} \neq 0), \quad N_\alpha(0) = 0, \quad \mathcal{B}_\alpha = \{\mathbf{v} : N_\alpha(\mathbf{v}) \leq 1\}.$$

The function N_α is finite and positive away from the origin. The scaling rule $h_\alpha(sW + b) = h_\alpha(W) + \log |s|$ for $s \neq 0$ makes it even and positively 1-homogeneous. To be a norm, it must therefore satisfy the triangle inequality

$$N_\alpha(\mathbf{v} + \mathbf{w}) \leq N_\alpha(\mathbf{v}) + N_\alpha(\mathbf{w}),$$

or equivalently be convex. This is known to hold for every planar convex body K , without a symmetry assumption, and every $0 \leq \alpha \leq \infty$ [5, 6, 7].

We study this convexity through angular derivatives. Write $\mathbf{u}_\theta = (\cos \theta, \sin \theta)$, $\mathbf{u}_\theta^\perp = (-\sin \theta, \cos \theta)$, and

$$H_\alpha(\theta) = h_\alpha(\langle \mathbf{u}_\theta, \mathbf{X} \rangle), \quad r_\alpha(\theta) = e^{H_\alpha(\theta)}.$$

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I used LLMs to assist with analysis and synthesis in the preparation of this article, particularly GPT-5.6 Sol, GPT-6 Astra, and Claude Fable 5.1 (all accessed in part via Poe with the support of Quora, where I am an advisor). In particular, the focal covariance identity in this article was discovered in dialogue with GPT-5.6 Sol while exploring potential routes towards the Ball–Nayar–Tkocz norm conjecture [2]. The problem, final methods, and final written form are my own; and of course any errors remain my responsibility. This work was conducted while I was visiting the Technological Innovation, Entrepreneurship, and Strategic Management (TIES) Group at the MIT Sloan School of Management; I greatly appreciate their hospitality.

Then $N_\alpha(\mathbf{v}) = |\mathbf{v}|r_\alpha(\arg \mathbf{v})$, and $\mathcal{B}_\alpha$ has radial function $1/r_\alpha$. By the support-function criterion, convexity of N_α is equivalent to nonnegativity of the distribution $r_\alpha + r''_\alpha$ on the circle [14]. For polygons and $0 < \alpha < \infty$, we compute this distribution explicitly from the boundary slopes and the lengths of the supporting edges.

For a unit vector $\mathbf{u}$, let $\ell_{\mathbf{u}}(z)$ be the length of the chord $K \cap \{\mathbf{x} : \langle \mathbf{u}, \mathbf{x} \rangle = z\}$. Thus the projection $\langle \mathbf{u}, \mathbf{X} \rangle$ has density $\ell_{\mathbf{u}}/|K|$. For a polygon K , *edge-normal directions* mean both perpendicular directions to each edge, with repeated directions counted only once. Away from these directions, use coordinates $z = \langle \mathbf{u}_\theta, \mathbf{x} \rangle$ and $d = \langle \mathbf{u}_\theta^\perp, \mathbf{x} \rangle$ to write

$$(1.1) \quad K = \{(z, d) : a_- \leq z \leq a_+, k(z) \leq d \leq g(z)\},$$

where the upper and lower boundary functions g, k are concave and convex, respectively. In this direction, write $\ell = \ell_{\mathbf{u}_\theta} = g - k$ and $c = (g + k)/2$ for the chord length and the height of its midpoint. We have $\ell(a_\pm) = 0$. For $0 < \alpha < \infty$, let Z_α have the *escort density* $\ell^\alpha / \int \ell^\alpha$; at $\alpha = 1$ this is the marginal density itself. Primes on H_α, r_α denote angular derivatives; those on g, k, ℓ, c denote derivatives in z , with slopes understood almost everywhere.

We show the following covariance identity, which to our knowledge is new.

Theorem 1.1. *Let K be a convex polygon and $0 < \alpha < \infty$. Away from the directions normal to its edges, H_α is C^2 , and*

$$(1.2) \quad H'_\alpha(\theta) = \mathbb{E}c'(Z_\alpha) = \mathbb{E}g'(Z_\alpha) = \mathbb{E}k'(Z_\alpha),$$

$$(1.3) \quad \frac{r_\alpha + r''_\alpha}{r_\alpha}(\theta) = -\alpha \text{Cov}(g'(Z_\alpha), k'(Z_\alpha)).$$

The right side is nonnegative, and it is positive exactly when neither boundary chain is a single edge.

The slopes g' and k' are oppositely monotone, so their covariance is nonpositive. Thus the convexity criterion holds between edge-normal directions. To complete it on the circle, we must also determine the derivative jumps at those directions. At an edge normal θ_0 , let $L_\pm$ be the lengths of the supporting faces with outer normals $\pm \mathbf{u}_{\theta_0}$, with a vertex assigned length 0. We prove in Theorem 3.2 that

$$H'_\alpha(\theta_0+) - H'_\alpha(\theta_0-) = \frac{L_+^{\alpha+1} + L_-^{\alpha+1}}{(\alpha+1) \int \ell_{\mathbf{u}_{\theta_0}}^\alpha dz} \quad (0 < \alpha < \infty);$$

multiplying this positive jump by $r_\alpha(\theta_0)$ gives the mass of the atom of $r_\alpha + r''_\alpha$ at θ_0 . Together with the covariance density, these atoms give the complete measure decomposition of $r_\alpha + r''_\alpha$ for polygons (Proposition 3.3). Both parts are nonnegative, so N_α is convex for polygons. Continuity under Hausdorff approximation gives convexity for arbitrary planar convex bodies, and the endpoint orders follow by taking limits. This yields a unified proof of the norm property of N_α for every planar convex body, without a symmetry assumption, throughout the range $0 \leq \alpha \leq \infty$ (Theorem 3.4). For finite positive orders, this property is known, and follows from the convexity theorems of Gardner and Zhang [6] and Haddad [7], through the relation between planar cross-section bodies and radial mean bodies established by Gardner and Giannopoulos [5], as explained below. The endpoint orders also admit elementary proofs using difference bodies (Remark 4.4).

The equality case also determines the boundary geometry of $\mathcal{B}_\alpha$. For polygons, zero covariance on an interval gives a straight boundary segment, strictly negative covariance gives an arc of positive curvature, and the derivative jumps give corners (Proposition 4.1). To retain the equality information for bodies that are not polygons, we prove a one-sided measure inequality under Hausdorff approximation (Proposition 4.2), which yields the following characterization.

Theorem 1.2. *Let K be a planar convex body and $0 < \alpha < \infty$. Then $\mathcal{B}_\alpha$ is a polygon if and only if K is a triangle. In that case, $\mathcal{B}_\alpha$ is a centrally symmetric hexagon.*

For a triangle, one boundary chain is a single edge in every direction away from edge normals, so the covariance vanishes on each of the six intervening arcs. The resulting hexagon and its explicit formulas are already given by the simplex equality case in [6, Theorem 5.5], or directly by [5, Corollary 6.2]; we recover them in Section 4. We have not found the converse or the exact edge-normal jump formula in the cited literature.

The restriction to finite positive orders is essential in Theorem 1.2: at orders 0 and ∞ , every polygon has a polygonal associated unit ball (Remark 4.4).

At finite positive orders, the unit balls $\mathcal{B}_\alpha$ are normalized versions of the *p-cross-section bodies* introduced by Gardner and Giannopoulos [5]. For $p > -1$, the radial function of $C_p K$ is the p -th mean of the length of the chord perpendicular to the given direction through a uniform point of K ; the mean is geometric at $p = 0$. Writing $\rho_{C_p K}$ for its radial function, integration along chords gives

$$\rho_{C_p K}(\mathbf{u}) = \left(\frac{1}{|K|} \int_{\mathbb{R}} \ell_{\mathbf{u}}^{p+1} \right)^{1/p}, \quad e^{-h_{p+1}(\langle \mathbf{u}, \mathbf{X} \rangle)} = \frac{\rho_{C_p K}(\mathbf{u})}{|K|} \quad (p \neq 0).$$

Taking the limit at $p = 0$ gives the geometric mean and Shannon entropy, so

$$(1.4) \quad \mathcal{B}_\alpha = |K|^{-1} C_{\alpha-1} K \quad (0 < \alpha < \infty).$$

This identifies convexity of N_α with convexity of the corresponding planar cross-section body. Gardner and Giannopoulos identified these bodies with rotated dilates of radial mean bodies [5, Section 5]. For finite $\alpha > 1$, convexity follows from Gardner and Zhang's radial-mean-body theorem [6, Theorem 4.3]; the Shannon case $\alpha = 1$ follows by continuity. Haddad proved the remaining orders $0 < \alpha < 1$ by a polygonal argument [7, Theorem 1]; Langharst subsequently proved convexity of radial mean bodies for every $p > -1$ in all dimensions [9, Theorem 3]. The endpoint orders have elementary descriptions (Remark 4.4); see also [4] for the geometric background.

Haddad's polygonal proof already combines local convexity with derivative jumps, after translating from radial-mean-body directions to projection directions by a rotation through $\pi/2$. For $0 < \alpha < 1$, he expresses a power of the radial-mean norm as a signed sum of powers of linear forms; a sign rule gives convexity on each cone [7, Propositions 6, 7, and 9]. For generic polygonal configurations, the pieces have upward derivative jumps at edge normals and match in C^1 at other vertex-projection coincidences [7, Propositions 11 and 12]. Our formula determines the second variation and its equality case at every finite positive order. It also gives C^2 regularity at the non-edge coincidences and the exact jump magnitude, including when both supporting faces are edges.

For nonuniform densities, convexity of the same entropy functionals is part of a broader conjecture. Ball, Nayar, and Tkocz conjectured that $\mathbf{v} \mapsto \exp h_1(\langle \mathbf{v}, \mathbf{X} \rangle)$, extended by 0 at the origin, is a norm for every symmetric log-concave random vector with a density; they proved subadditivity of its $(1/5)$ -th power [2]. Madiman, Melbourne, and Xu subsequently stated a Rényi family of subadditivity exponents, with Shannon exponent approximately 0.240789 [12, Theorem 3.20 and Remark 3.21]. Kominers [8, Theorem 1.1] proved subadditivity of N_1^κ for every $0 < \kappa \leq 9/16$, and hence of the square root of the Shannon functional. Madiman, Melbourne, and Xu formulated corresponding norm conjectures for orders $\alpha \in [1, \infty]$ for symmetric log-concave random vectors, and more generally symmetric κ -concave random vectors, whenever the functional is finite everywhere [12, Conjecture 3.22]. Li proved convexity at order 2 for log-concave densities without a symmetry assumption [11, Theorem 3.3].

At order 0, the functional is the width of the convex support K , namely h_{K-K} , and hence is a finite-valued norm when K is bounded; cf. [11, Theorem 3.1 and Remark 3.9] for the symmetric case. At order ∞ , the norm property for uniform measures on origin-symmetric convex bodies follows from Busemann's intersection-body theorem [3]; for symmetric log-concave densities, it follows from Ball's extension [1]; see also the proof of [2, Proposition 1] for the reciprocal-marginal-maximum formulation. The Shannon conjecture reduces to planar densities by taking 2-dimensional marginals, which preserve log-concavity [13] but need not preserve uniformity. The argument here uses the uniform conditional law along each chord and does not give the required bound for general log-concave densities (Section 5.4).

The proof of Theorem 1.1 works directly with chord-length integrals. A small change in projection direction can be represented, up to a scalar factor, by the shear $(z, d) \mapsto (z + td, d)$. We label each sheared chord by the original abscissa of its upper endpoint, so the integration interval stays fixed. The change of variables introduces the upper boundary slope. At a fixed label, the derivative of a chord length is minus that length times the lower boundary slope. The product of the two slopes therefore enters the second variation and leads to the covariance formula. Away from edge normals the slopes are bounded, so the differentiated integrals remain integrable at vanishing chords for every finite positive Rényi order. Section 2 carries out the calculation, and Section 3 computes the edge-normal

limits needed to complete the convexity proof. Section 4 studies the boundary geometry of the unit balls and proves the triangle characterization. Section 5 gives examples and discusses regularity and the limitations of the method.

2. THE COVARIANCE IDENTITY

Let K be a convex polygon. Fix $0 < \alpha < \infty$ and a direction $\mathbf{u}_{\theta_0}$ not normal to an edge, and use the chord coordinates in (1.1). The corresponding coordinates of the uniform random point $\mathbf{X}$ are the scalar random variables

$$Z = \langle \mathbf{u}_{\theta_0}, \mathbf{X} \rangle, \quad D = \langle \mathbf{u}_{\theta_0}^\perp, \mathbf{X} \rangle.$$

We write $A = g'$ and $B = k'$; these slopes are bounded step functions. Dots denote differentiation in the shear parameter t .

2.1. Angles and shears. We replace rotation of the projection direction by the shear $Z + tD$. Since $\mathbf{u}_{\theta_0} + t\mathbf{u}_{\theta_0}^\perp = \sqrt{1+t^2} \mathbf{u}_{\theta_0 + \arctan t}$, the scaling rule gives, for $\eta_\alpha(t) = h_\alpha(Z + tD)$,

$$(2.1) \quad \eta_\alpha(t) = \frac{1}{2} \log(1+t^2) + H_\alpha(\theta_0 + \arctan t).$$

Thus the one-sided first derivatives at 0 and θ_0 agree whenever they exist, and, where H_α is twice differentiable,

$$\dot{\eta}_\alpha(0) = H'_\alpha(\theta_0), \quad \ddot{\eta}_\alpha(0) = 1 + H''_\alpha(\theta_0).$$

Consequently,

$$(2.2) \quad \ddot{\eta}_\alpha(0) + \dot{\eta}_\alpha(0)^2 = \frac{r_\alpha + r''_\alpha}{r_\alpha}(\theta_0).$$

Choose $t_0 > 0$ with $t_0 \max(\|A\|_\infty, \|B\|_\infty) < 1$. For $|t| < t_0$, the sheared polygon $K_t = \{(z + td, d) : (z, d) \in K\}$ has no vertical edge, and $(Z + tD, D)$ is uniform on K_t . Use subscripts t for its boundary functions, slopes, chord lengths, and support endpoints. Shears preserve area, so $\int \ell_t = |K|$.

2.2. The variation formulas. We compute the entropy derivatives from the chord-length integrals

$$I(t) = \int \ell_t^\alpha dz, \quad J(t) = \int \ell_t^\alpha c'_t dz,$$

with integrals over $[a_-(t), a_+(t)]$. The integral I determines the Rényi entropy for $\alpha \neq 1$, and J/I will give its first variation.

Lemma 2.1. *For $|t| < t_0$, the functions I and J are C^1 , $I > 0$,*

$$(2.3) \quad \dot{I} = (1 - \alpha)J, \quad \dot{J} = -\alpha \int \ell_t^\alpha A_t B_t dz,$$

and η_α is C^2 with $\dot{\eta}_\alpha = J/I$.

Proof. First, the weighted slope averages agree: we have

$$\int \ell_t^\alpha (A_t - B_t) dz = \frac{1}{\alpha + 1} [\ell_t^{\alpha+1}]_{a_-(t)}^{a_+(t)} = 0$$

because the chord length vanishes at both endpoints. The chain rule applies since ℓ_t is Lipschitz and $y \mapsto y^{\alpha+1}$ is C^1 on its range. Since $c'_t = (A_t + B_t)/2$, we have

$$(2.4) \quad J(t) = \int \ell_t^\alpha A_t dz = \int \ell_t^\alpha B_t dz.$$

A fixed integration interval. Follow a point $(x, g(x))$ on the original upper boundary. Its abscissa after the shear is $\Phi_t(x) = x + tg(x)$, and the chord below it has length $\hat{\ell}_t(x) = \ell_t(\Phi_t(x))$. The map Φ_t is increasing from $[a_-, a_+]$ onto $[a_-(t), a_+(t)]$, with derivative $1 + tA(x) > 0$ almost everywhere, and

$$g_t(\Phi_t(x)) = g(x), \quad A_t(\Phi_t(x)) = \frac{A(x)}{1 + tA(x)}.$$

Let $d = b_j + \beta_j z$ be the lines containing the lower edges. The lower boundary is their maximum: $k(z) = \max_j (b_j + \beta_j z)$. After the shear these lines become $d = (b_j + \beta_j z)/(1 + t\beta_j)$, so

$$(2.5) \quad \begin{aligned} \widehat{\ell}_t(x) &= \min_j \frac{g(x) - b_j - \beta_j x}{1 + t\beta_j}, \\ \partial_t \widehat{\ell}_t(x) &= -\widehat{\ell}_t(x) B_t(\Phi_t(x)). \end{aligned}$$

The derivative formula follows by differentiating the term corresponding to the lower endpoint's edge; it holds whenever that endpoint is not a vertex. Changing variables by Φ_t now gives integrals on the fixed interval:

$$I(t) = \int_{a_-}^{a_+} (1 + tA) \widehat{\ell}_t^\alpha dx, \quad J(t) = \int_{a_-}^{a_+} A \widehat{\ell}_t^\alpha dx.$$

Differentiation. Work on a compact subinterval of $(-t_0, t_0)$. The chord lengths and slopes are uniformly bounded there, and every denominator in (2.5) is bounded away from 0. Each term in the minimum is therefore uniformly Lipschitz in t , as is their minimum. For each fixed interior x , every numerator is at least $\ell(x) > 0$, so the chord length stays bounded away from 0 on that interval. Its power and its entropy integrand are therefore absolutely continuous in t .

At a differentiability point, a finite minimum has the derivative of any active term. The chain rule and (2.5) therefore give, almost everywhere in t ,

$$\begin{aligned} \partial_t(\widehat{\ell}_t^\alpha) &= -\alpha \widehat{\ell}_t^\alpha B_t(\Phi_t(x)), \\ \partial_t(\widehat{\ell}_t \log \widehat{\ell}_t) &= -\widehat{\ell}_t(1 + \log \widehat{\ell}_t) B_t(\Phi_t(x)). \end{aligned}$$

Both right-hand sides extend by 0 at a vanishing chord and are uniformly bounded in x, t , since y^α and $y|1 + \log y|$ are bounded on bounded intervals. At $x = a_\pm$, the chord length is identically 0. Absolute continuity and these derivative bounds give uniformly bounded difference quotients even when the lower endpoint changes edges. In particular, vanishing chords cause no singularity for $0 < \alpha < 1$.

At each fixed t , the lower endpoint is a vertex for only finitely many labels x . Outside that set the displayed derivatives exist, so dominated convergence permits differentiation under the integral. Differentiating I, J and changing back to z yields

$$(2.6) \quad \dot{I} = \int \ell_t^\alpha (A_t - \alpha B_t) dz = (1 - \alpha)J, \quad \dot{J} = -\alpha \int \ell_t^\alpha A_t B_t dz.$$

The derivatives in (2.6) are also continuous: at a fixed t , outside the same finite set of labels, the lower endpoint stays on the same edge under a sufficiently small change of t . The transformed integrands therefore converge almost everywhere, and their uniform bounds allow another application of dominated convergence. Hence I and J are C^1 .

The entropy. For $\alpha \neq 1$, $\eta_\alpha = (\log I - \alpha \log |K|)/(1 - \alpha)$, so $\dot{\eta}_\alpha = J/I$. At $\alpha = 1$, the same change of variables gives

$$\begin{aligned} \frac{d}{dt} \int \ell_t \log \ell_t dz &= \int \ell_t \log \ell_t (A_t - B_t) dz - \int \ell_t B_t dz \\ &= \left[\frac{\ell_t^2}{2} \log \ell_t - \frac{\ell_t^2}{4} \right]_{a_-(t)}^{a_+(t)} - J \\ &= -J. \end{aligned}$$

The boundary term is 0 because $\ell_t = 0$ at the endpoints and $y^2 \log y \rightarrow 0$ as $y \downarrow 0$. Since $\eta_1 = \log |K| - |K|^{-1} \int \ell_t \log \ell_t$ and $I = |K|$, again $\dot{\eta}_1 = J/I$. As $I > 0$ and I, J are C^1 , every η_α is C^2 . $\square$

Under the shear, a point at height d moves horizontally with velocity d . Conditional on $Z = z$, the mean velocity is $c(z)$. Thus the first variation $\dot{\eta}_\alpha(0) = \mathbb{E}c'(Z_\alpha)$ averages the spatial change in this mean velocity, with the usual chord-length weight at $\alpha = 1$.

Note that the argument for Lemma 2.1 includes coincidences between upper and lower projected vertices: these do not affect C^2 regularity away from edge normals, although C^3 regularity can fail there (Section 5.2).

2.3. Covariance and equality.

Proof of Theorem 1.1. By Lemma 2.1, η_α is C^2 near 0 and $\dot{\eta}_\alpha(0) = J/I = \mathbb{E}c'(Z_\alpha)$. Equation (2.4) gives the same average for both boundary slopes. By (2.3), at $t = 0$,

$$\ddot{\eta}_\alpha = \frac{\dot{J}}{I} - \frac{J\dot{I}}{I^2} = -\alpha \mathbb{E}[A(Z_\alpha)B(Z_\alpha)] + (\alpha - 1)\dot{\eta}_\alpha^2.$$

Hence

$$\ddot{\eta}_\alpha + \dot{\eta}_\alpha^2 = -\alpha \left(\mathbb{E}[A(Z_\alpha)B(Z_\alpha)] - \mathbb{E}A(Z_\alpha)\mathbb{E}B(Z_\alpha) \right) = -\alpha \text{Cov}(A(Z_\alpha), B(Z_\alpha)),$$

and (2.1)–(2.2) give (1.2) and (1.3).

For the sign and equality case, we use a step-function form of Hoeffding's covariance identity [10]. Expand the two slope functions at their jumps. Let $\mathcal{V}_+$ and $\mathcal{V}_-$ be the vertices of the upper and lower chains other than the two extreme vertices at $a_\pm$, omitting redundant collinear points. Write z_v for a vertex abscissa, $[A]_v < 0$ and $[B]_w > 0$ for the respective slope jumps, and F_α for the distribution function of Z_α . Then

$$(2.7) \quad -\text{Cov}(A(Z_\alpha), B(Z_\alpha)) = \sum_{v \in \mathcal{V}_+} \sum_{w \in \mathcal{V}_-} |[A]_v| |[B]_w| F_\alpha(\min(z_v, z_w))(1 - F_\alpha(\max(z_v, z_w))).$$

Indeed, expand $A(z) = A(a_- +) + \sum_{v \in \mathcal{V}_+} [A]_v \mathbf{1}_{\{z > z_v\}}$ and the analogous expression for B , and use

$$\text{Cov}(\mathbf{1}_{\{Z_\alpha > s\}}, \mathbf{1}_{\{Z_\alpha > t\}}) = F_\alpha(\min(s, t))(1 - F_\alpha(\max(s, t))).$$

Every summand in (2.7) is positive, since all the vertex abscissae are interior to the support. The sum is empty exactly when one boundary chain is a single edge, proving the final assertion of the theorem. $\square$

Each term in (2.7) multiplies the magnitudes of two slope jumps by the escort probabilities of lying to the left of both vertices and to the right of both. Thus slope changes on opposite chains need not occur at the same abscissa to contribute positively to $r_\alpha + r''_\alpha$. Figure 1(a) illustrates these weights at Shannon order, when the probabilities are area fractions.

3. EDGE NORMALS AND CONVEXITY

3.1. Continuity under approximation. We next record the continuity needed both at edge normals and in the passage from polygons to arbitrary bodies. Write $h_K(\mathbf{v}) = \max_{\mathbf{x} \in K} \langle \mathbf{v}, \mathbf{x} \rangle$ for the *support function* of K .

Lemma 3.1. *Let $0 < \alpha < \infty$.*

- (a) *Let $K_j \rightarrow K$ be convex bodies converging in the Hausdorff metric, let $\mathbf{X}_j$ and $\mathbf{X}$ be uniform on K_j and K , let $\mathbf{u}$ be a unit vector, and let ℓ_j and ℓ be the chord-length functions of K_j and K for sections perpendicular to $\mathbf{u}$, extended by zero. Then $\ell_j \rightarrow \ell$ almost everywhere, the functions ℓ_j are uniformly bounded and supported in a common compact interval, and $h_\alpha(\langle \mathbf{u}, \mathbf{X}_j \rangle) \rightarrow h_\alpha(\langle \mathbf{u}, \mathbf{X} \rangle)$.*
- (b) *For every convex body K , the function N_α is continuous on $\mathbb{R}^2$.*

Proof. (a) Translating all bodies by a common vector only shifts the chord-length functions and does not change entropies, so we may assume that K contains a disk of radius $\varrho > 0$ centered at the origin. Let $\varepsilon_j = d_H(K_j, K)$ and $\tau_j = \varepsilon_j/\varrho$. On the unit circle $h_K \geq \varrho$, so $|h_{K_j} - h_K| \leq \varepsilon_j \leq \tau_j h_K$, and the support-function criterion for inclusion [14] gives $(1 - \tau_j)K \subset K_j \subset (1 + \tau_j)K$ for all large j . In particular $|K_j| \rightarrow |K|$, and

$$(1 - \tau_j)\ell\left(\frac{z}{1 - \tau_j}\right) \leq \ell_j(z) \leq (1 + \tau_j)\ell\left(\frac{z}{1 + \tau_j}\right).$$

Since ℓ is continuous in the interior of its support, $\ell_j \rightarrow \ell$ there, and outside the closed support the upper bound is eventually zero. No convergence is needed at the two support endpoints. A common enclosing disk bounds both chord lengths and supports. The marginal densities $f_j = \ell_j/|K_j|$ are therefore uniformly bounded, supported in a common compact interval, and converge almost everywhere

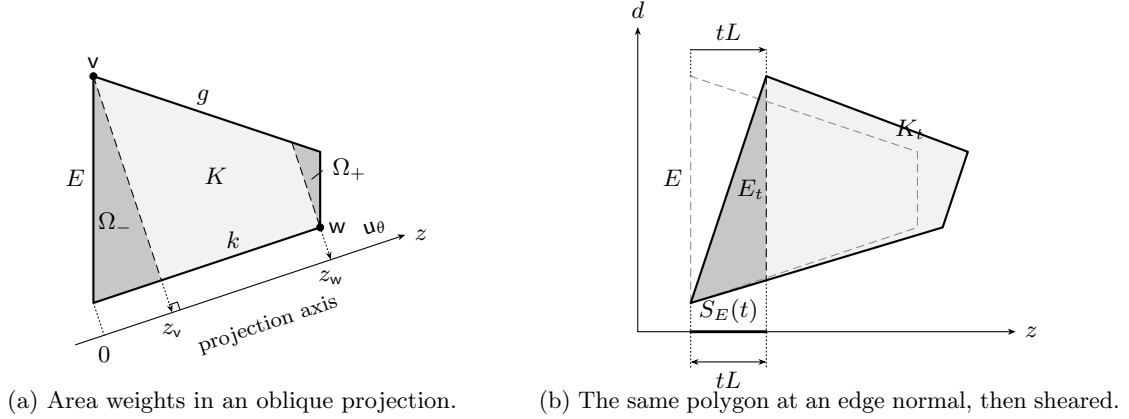


FIGURE 1. Covariance and edge-normal contributions to the angular second variation. (a) At Shannon order, the shaded area fractions weight the pair of boundary-slope jumps at the marked vertices. (b) Near an edge normal, an affine chord-length profile on a shrinking interval produces a finite derivative jump.

Both panels start from $K = \text{conv}\{(0, 0), (0, 3), (3, 2), (3, 1)\}$ in the same position and at the same scale. (a) Project in direction $u_\theta = (3, 1)/\sqrt{10}$. The marked vertices $v = (0, 3)$ and $w = (3, 1)$ lie on the upper and lower chains, with $z_v < z_w$. The dashed chords pass through them; dotted arrows locate their projections on the displaced axis, whose 0 is the perpendicular foot of the origin. The shaded regions Ω_- and Ω_+ consist of points with $z < z_v$ and $z > z_w$, respectively. Thus $|\Omega_-|/|K| = F_1(z_v)$ and $|\Omega_+|/|K| = 1 - F_1(z_w)$: their product multiplies $|[A]_v| |[B]_w|$ in (2.7). (b) Switch to horizontal projection, normal to the vertical edge E of length $L = 3$. The dashed outline is K ; the solid outline is its shear $(z, d) \mapsto (z + td, d)$, drawn at $t = 1/3$. The edge E_t has slope $1/t$ and projects onto $S_E(t)$, of width tL . The shaded triangle consists of the chords over this interval; their lengths increase affinely from 0 to $L + O(t)$ as $t \downarrow 0$.

to $f = \ell/|K|$. Since $s \mapsto s^\alpha$ and $s \mapsto s \log s$ are bounded on bounded subsets of $[0, \infty)$, dominated convergence gives $\int f_j^\alpha \rightarrow \int f^\alpha$ and $\int f_j \log f_j \rightarrow \int f \log f$, hence the convergence of entropies.

(b) Applying (a) to rotations of K shows that H_α is continuous. Hence $N_\alpha(v) = |v| r_\alpha(\arg v)$ is continuous on $\mathbb{R}^2 \setminus \{0\}$, and $N_\alpha(v) \leq |v| \max r_\alpha$ gives continuity at the origin. $\square$

3.2. The derivative jump. At an edge normal, a boundary slope becomes unbounded under a small shear, but only on a shrinking interval. The next formula records its finite contribution to the first variation.

Theorem 3.2. *Let X be uniform on a convex polygon K and $0 < \alpha < \infty$. Fix a direction u_{θ_0} , and let $L_\pm$ be the lengths of the faces with outer normals $\pm u_{\theta_0}$, a vertex having length 0 . Write $\ell_{u_{\theta_0}}$ for the chord-length function in this direction. Then H_α is continuous at θ_0 , its derivative has finite one-sided limits equal to its one-sided derivatives there, and*

$$(3.1) \quad H'_\alpha(\theta_0+) - H'_\alpha(\theta_0-) = \frac{L_+^{\alpha+1} + L_-^{\alpha+1}}{(\alpha+1) \int \ell_{u_{\theta_0}}^\alpha dz}.$$

For $\alpha = 1$ the denominator is $2|K|$, so the jump of H'_1 is $(L_+^2 + L_-^2)/(2|K|)$; if K is centrally symmetric, it is $L^2/|K|$, where L is the common length of the two faces.

Proof of Theorem 3.2. Use the coordinates $z = \langle u_{\theta_0}, x \rangle$, $d = \langle u_{\theta_0}^\perp, x \rangle$, and write $X = (Z, D)$. Let $[a_-, a_+]$ be the projected support, so $L_\pm = \ell(a_\pm)$. As in Section 2, put $\eta_\alpha(t) = h_\alpha(Z + tD)$. By (2.1), it suffices to compute the jump of $\dot{\eta}_\alpha$ at 0 . If $L_+ = L_- = 0$, then K has no vertical edge and there is nothing to prove, by Lemma 2.1. In general there is $t_0 > 0$ such that the sheared polygons K_t have no vertical edge for $0 < |t| < t_0$. Use $I(t) = \int \ell_t^\alpha$ and $J(t) = \int \ell_t^\alpha c'_t$ as in Lemma 2.1. Applying Lemma 2.1 around each such t shows that η_α is C^2 on $(-t_0, 0) \cup (0, t_0)$ with $\dot{\eta}_\alpha = J/I$. By Lemma 3.1(a), applied to $K_t \rightarrow K$, the function η_α is continuous at 0 and $I(t) \rightarrow \int \ell^\alpha$. It remains to find the one-sided limits of J .

Use the upper-chain representation (2.4):

$$J(t) = \sum_{e \text{ upper}} \sigma_e(t) \int_{S_e(t)} \ell_t^\alpha dz,$$

where $S_e(t)$ is the projection of the sheared edge e and $\sigma_e(t)$ is its slope. Each nonvertical edge keeps its chain membership for small $|t|$: its horizontal component $\Delta z + t\Delta d$ keeps its sign. Its slope and projected endpoints converge, and $\ell_t \rightarrow \ell$ almost everywhere with uniform bounds. Thus the nonvertical upper edges have a common total contribution J_* in both one-sided limits.

Let E be a vertical supporting edge of length $L > 0$. After shear, it has slope $1/t$ and a projection $S_E(t)$ of length $|t|L$. For sufficiently small nonzero t , one endpoint is the leftmost or rightmost point of K_t , and the opposite boundary over $S_E(t)$ is the nonvertical edge incident to that endpoint. Indeed, the remaining vertices have projections separated from this supporting line at $t = 0$, so none enters the shrinking interval $S_E(t)$ for small t . Consequently ℓ_t is affine on $S_E(t)$, vanishes at the extreme endpoint, and has height $M_E(t) \rightarrow L$ at the other endpoint. The last limit follows because the incident edge has bounded slope and $|S_E(t)| \rightarrow 0$; see Figure 1(b). Whenever E belongs to the upper chain, its contribution is exactly

$$\frac{1}{t} \int_{S_E(t)} \ell_t^\alpha dz = \operatorname{sgn}(t) \frac{L M_E(t)^\alpha}{\alpha + 1}.$$

The left supporting edge belongs to the upper chain for $t > 0$, and the right one for $t < 0$. Therefore

$$J(0+) = J_* + \frac{L_-^{\alpha+1}}{\alpha+1}, \quad J(0-) = J_* - \frac{L_+^{\alpha+1}}{\alpha+1},$$

with a missing edge contributing 0. Dividing their difference by $I(0) = \int \ell^\alpha$ proves (3.1) for the limits of η_α . Continuity and the mean value theorem identify these limits with the one-sided derivatives of η_α at 0. Equation (2.1) transfers the conclusion to H_α . $\square$

3.3. Convexity. For a continuous 2π -periodic function r , the planar support-function criterion states that $\mathbf{v} \mapsto |\mathbf{v}|r(\arg \mathbf{v})$, extended by 0 at the origin, is convex if and only if $r + r''$ is a nonnegative distribution on the circle [14]. Equivalently,

$$\int_0^{2\pi} r(\theta)(\varphi(\theta) + \varphi''(\theta)) d\theta \geq 0$$

for every nonnegative smooth 2π -periodic test function φ . We apply this criterion to r_α by combining the covariance identity and the derivative-jump formula.

Fix a polygon K and $0 < \alpha < \infty$. Write $\theta_1 < \dots < \theta_m$ for its distinct edge-normal directions in $[0, 2\pi)$. Let $\Delta_j > 0$ be the jump of H'_α at θ_j given by Theorem 3.2, and let δ_θ denote unit mass at θ .

Proposition 3.3. *As distributions on the circle,*

$$r_\alpha + r''_\alpha = -\alpha r_\alpha \operatorname{Cov}(g'(Z_\alpha), k'(Z_\alpha)) d\theta + \sum_{j=1}^m r_\alpha(\theta_j) \Delta_j \delta_{\theta_j},$$

where the density is evaluated away from the edge normals. In particular, $r_\alpha + r''_\alpha$ is a nonnegative measure.

Proof. On each arc (a, b) between consecutive θ_j , Theorem 1.1 gives $r_\alpha \in C^2$ and $r_\alpha + r''_\alpha \geq 0$. Thus $r''_\alpha \geq -C$, where $C = \max r_\alpha$. The one-sided limits of $r'_\alpha = r_\alpha H'_\alpha$ at a and b are finite by Theorem 3.2. Consequently,

$$\int_a^b (r''_\alpha + C) d\theta = r'_\alpha(b-) - r'_\alpha(a+) + C(b-a) < \infty,$$

so $r''_\alpha \in L^1(a, b)$. For a smooth test function φ on the circle, integration by parts on each arc gives

$$\begin{aligned} \int r_\alpha(\varphi + \varphi'') d\theta &= \sum_{(a,b)} \left\{ \int_a^b (r_\alpha + r''_\alpha) \varphi d\theta + [r_\alpha \varphi' - r'_\alpha \varphi]_{a+}^{b-} \right\} \\ &= -\alpha \int r_\alpha \operatorname{Cov}(g'(Z_\alpha), k'(Z_\alpha)) \varphi d\theta + \sum_j [r'_\alpha]_{\theta_j} \varphi(\theta_j). \end{aligned}$$

At each θ_j , Theorem 3.2 gives continuity of H_α , hence of $r_\alpha = e^{H_\alpha}$. Thus the terms $r_\alpha \varphi'$ cancel, and taking one-sided limits in $r'_\alpha = r'_\alpha H'_\alpha$ gives

$$[r'_\alpha]_{\theta_j} = r_\alpha(\theta_j) [H'_\alpha]_{\theta_j} = r_\alpha(\theta_j) \Delta_j. \quad \square$$

Proposition 3.3, together with Hausdorff approximation and limits in the Rényi order, gives a new proof of the following norm theorem for uniform measures on arbitrary planar convex bodies.

Theorem 3.4 ([5, 6, 7]). *Let X be uniform on a convex body $K \subset \mathbb{R}^2$. For every $0 \leq \alpha \leq \infty$, N_α is a norm.*

Proof. Proposition 3.3 and the support-function criterion prove convexity for polygons and $0 < \alpha < \infty$. For a general convex body choose polygons $K_j \rightarrow K$ in the Hausdorff metric, and let $N_{\alpha,j}$ denote the functional for a uniform point of K_j . Lemma 3.1 and homogeneity give pointwise convergence $N_{\alpha,j} \rightarrow N_\alpha$, so convexity passes to the limit. Positivity and even homogeneity then make N_α a norm.

Finally, for a bounded compactly supported density f , $\int f^\alpha \rightarrow |\{f > 0\}|$ as $\alpha \downarrow 0$ by dominated convergence, while $\|f\|_\alpha \rightarrow \|f\|_\infty$ as $\alpha \rightarrow \infty$. Hence $N_\alpha \rightarrow N_0, N_\infty$ pointwise at the endpoints, and convexity passes to both limits. Each nonzero projection has finite positive support length and finite positive essential supremum of its density, so the limits are finite and strictly positive away from the origin. They are therefore norms as well; Remark 4.4 identifies their balls. $\square$

4. GEOMETRY OF THE UNIT BALLS

Now fix a polygon K and $0 < \alpha < \infty$. By Theorem 3.4, $\mathcal{B}_\alpha$ is a convex body with the origin as an interior point and center of symmetry. Its *polar body* is

$$\mathcal{B}_\alpha^\circ = \{y \in \mathbb{R}^2 : \langle y, v \rangle \leq 1 \text{ for every } v \in \mathcal{B}_\alpha\};$$

this is the *polar unit ball* of N_α , or equivalently the unit ball of the dual norm. Since N_α is the *gauge* $v \mapsto \inf\{s > 0 : v \in s\mathcal{B}_\alpha\}$ of $\mathcal{B}_\alpha$, it is the support function of $\mathcal{B}_\alpha^\circ$ [14]. In terms of support and radial functions, we have

$$h_{\mathcal{B}_\alpha^\circ}(u_\theta) = r_\alpha(\theta), \quad \rho_{\mathcal{B}_\alpha}(u_\theta) = \frac{1}{r_\alpha(\theta)}.$$

Here u_θ labels a normal direction of the polar unit ball and a radial direction of the unit ball. Thus a supporting line of $\mathcal{B}_\alpha^\circ$ at distance $r_\alpha(\theta)$ from the origin determines the radial segment $[0, u_\theta/r_\alpha(\theta)]$ in $\mathcal{B}_\alpha$. Figure 2 in Section 5.1 illustrates the construction.

The measure $r_\alpha + r''_\alpha$ in Proposition 3.3 is the *surface area measure* of $\mathcal{B}_\alpha^\circ$: it records boundary length by outer normal direction [14]. To see its meaning directly, let θ lie between edge normals and let $y \in \partial\mathcal{B}_\alpha^\circ$ have outer normal u_θ . Then $\langle y, u_\theta \rangle = r_\alpha(\theta)$, and $r_\alpha(\psi) - \langle y, u_\psi \rangle$ attains its minimum 0 at $\psi = \theta$. Differentiating there gives $\langle y, u_\theta^\perp \rangle = r'_\alpha(\theta)$. Hence y is unique, and these points trace the boundary of the polar unit ball as

$$\gamma(\theta) = r_\alpha(\theta)u_\theta + r'_\alpha(\theta)u_\theta^\perp, \quad \gamma'(\theta) = (r_\alpha + r''_\alpha)(\theta)u_\theta^\perp.$$

Where $r_\alpha + r''_\alpha > 0$, arc-length reparametrization gives a C^2 boundary curve with radius of curvature $r_\alpha + r''_\alpha$. Where the density vanishes on an interval, γ is constant and describes a vertex. At θ_j , the one-sided supporting points differ by $[r'_\alpha]_{\theta_j}u_{\theta_j}^\perp$, so the polar unit ball has a side of length $r_\alpha(\theta_j)\Delta_j$ with outer normal u_{θ_j} . Under polarity, these sides correspond to corners of $\mathcal{B}_\alpha$, and vertices of the polar unit ball correspond to segments in $\partial\mathcal{B}_\alpha$.

Proposition 4.1. *For a polygon K and $0 < \alpha < \infty$, let $\theta_1, \dots, \theta_m$ be the distinct edge-normal directions of K .*

- (a) The curve $\partial\mathcal{B}_\alpha$ is C^2 except at the points $\mathbf{u}_{\theta_j}/r_\alpha(\theta_j)$, where it has corners.
- (b) Between consecutive edge normals, the vertices $\mathbf{v}_+$ and $\mathbf{v}_-$ where $\langle \mathbf{u}_\theta, \cdot \rangle$ attains its maximum and minimum are fixed. If they are adjacent, N_α is linear on the corresponding open cone, and the corresponding arc of $\partial\mathcal{B}_\alpha$ is a straight segment. Otherwise this arc has strictly positive curvature.

Proof. (a) The radial parametrization $\mathbf{b}(\theta) = \mathbf{u}_\theta/r_\alpha(\theta)$ has derivative

$$\mathbf{b}'(\theta) = \frac{r_\alpha(\theta)\mathbf{u}_\theta^\perp - r'_\alpha(\theta)\mathbf{u}_\theta}{r_\alpha(\theta)^2}.$$

Between edge normals it is C^2 and regular, since the $\mathbf{u}_\theta^\perp$ component of $\mathbf{b}'$ is $1/r_\alpha > 0$. At θ_j , the one-sided tangents have that same nonzero component, but their $\mathbf{u}_{\theta_j}$ components differ by $-[r'_\alpha]_{\theta_j}/r_\alpha(\theta_j)^2 \neq 0$. They are therefore not parallel, giving a corner.

(b) The extreme vertices change only when $\mathbf{u}_\theta$ crosses an edge normal. Between successive normals, the two boundary chains run from $\mathbf{v}_-$ to $\mathbf{v}_+$, and one is a single edge exactly when these vertices are adjacent. Theorem 1.1 gives $r_\alpha + r''_\alpha = 0$ throughout the arc in that case, and $r_\alpha + r''_\alpha > 0$ otherwise. In the first case γ is constant, say equal to $\mathbf{y}$, so $r_\alpha(\theta) = \langle \mathbf{y}, \mathbf{u}_\theta \rangle$ and $N_\alpha(\mathbf{v}) = \langle \mathbf{y}, \mathbf{v} \rangle$ on the open cone. The points $\mathbf{u}_\theta/r_\alpha(\theta)$ therefore trace an open segment on $\{\mathbf{v} : \langle \mathbf{y}, \mathbf{v} \rangle = 1\}$. In the second case, differentiating $\mathbf{b}'$ and using $r'_\alpha = r_\alpha H'_\alpha$ gives the curvature

$$\begin{aligned} \frac{\det(\mathbf{b}', \mathbf{b}'')}{|\mathbf{b}'|^3} &= \frac{(r_\alpha + r''_\alpha)/r_\alpha^3}{(1 + (H'_\alpha)^2)^{3/2}/r_\alpha^3} \\ &= \frac{r_\alpha + r''_\alpha}{(1 + (H'_\alpha)^2)^{3/2}} = \frac{-\alpha r_\alpha \operatorname{Cov}(g'(Z_\alpha), k'(Z_\alpha))}{(1 + (H'_\alpha)^2)^{3/2}} > 0, \end{aligned}$$

where the last equality and strict positivity follow from Theorem 1.1. $\square$

For a fixed polygon K , the directions of corners and the flat and curved sectors are therefore independent of the finite positive order α . If the polygon K is centrally symmetric, then its extreme vertices in a nonexceptional direction are opposite and hence nonadjacent. In this case $\mathcal{B}_\alpha$ is strictly convex, although it has corners at the edge-normal directions.

To derive the converse half of Theorem 1.2, we pass back to an arbitrary planar convex body K , keeping $0 < \alpha < \infty$. In each direction, as before, we use the boundary functions g, k on the interior $I = (a_-, a_+)$ of the projected support, and let F_α be the distribution function of the escort density $\ell^\alpha / \int \ell^\alpha$. The positive, locally finite slope measures $d(-g') = -D^2g$ and $dk' = D^2k$ record decreases of the upper-boundary slope and increases of the lower-boundary slope; D^2 denotes the distributional second derivative in z . We define a nonnegative extended-real-valued function Q_α by pairing these slope changes with the escort probabilities of lying to the left and to the right of both locations, as in (2.7):

$$(4.1) \quad Q_\alpha(\theta) = \iint_{I^2} F_\alpha(\min(s, t))(1 - F_\alpha(\max(s, t))) d(-g')(s) dk'(t).$$

For a polygon away from edge normals, the slope measures are finite sums of point masses at vertex abscissae, with masses $|[A]_\mathbf{v}|$ on the upper chain and $|[B]_\mathbf{w}|$ on the lower chain. Substitution gives exactly the finite sum in (2.7), hence $Q_\alpha = -\operatorname{Cov}(g'(Z_\alpha), k'(Z_\alpha))$ in these directions.

Proposition 4.2. *For every planar convex body K and $0 < \alpha < \infty$, the function Q_α defined in (4.1) is lower semicontinuous and integrable on the circle, and*

$$(4.2) \quad r_\alpha + r''_\alpha \geq \alpha r_\alpha Q_\alpha d\theta$$

as measures. Moreover, $Q_\alpha(\theta) = 0$ exactly when at least one boundary graph in the direction θ is affine.

Proof. *Lower semicontinuity.* Fix a direction $\mathbf{u}$ and let $K_j \rightarrow K$ in the Hausdorff metric. We first show that $Q_\alpha(K, \mathbf{u}) \leq \liminf_j Q_\alpha(K_j, \mathbf{u})$, writing the body and direction as arguments to distinguish these quantities. A common translation does not change Q_α , so we may place the origin in the interior

of K . The homothetic inclusions from Lemma 3.1 imply that every compact subinterval of I lies in the projected support of K_j for all sufficiently large j . On each such interval they give, with $\tau_j \rightarrow 0$,

$$(1 - \tau_j)g\left(\frac{z}{1 - \tau_j}\right) \leq g_j(z) \leq (1 + \tau_j)g\left(\frac{z}{1 + \tau_j}\right);$$

the analogous bounds for k_j have the inequalities reversed. Continuity of g and k makes the bounding functions converge uniformly on these intervals, so $g_j \rightarrow g$ and $k_j \rightarrow k$ locally uniformly on I .

This also gives convergence of the slope measures. For a smooth test function ψ supported in the interior of a compact subinterval $J \subset I$, distributional differentiation gives, for all large j ,

$$\int \psi d(-g'_j) = - \int_J g_j \psi'' dz \longrightarrow - \int_J g \psi'' dz = \int \psi d(-g').$$

To bound the mass on a compact subinterval, choose a nonnegative smooth ψ equal to 1 there and supported in a larger compact subinterval of I —its integrals converge and bound the mass, since the slope measures are positive. Uniform approximation by smooth functions supported in a fixed compact subinterval then extends the convergence to continuous test functions supported away from the endpoints. The same argument applies to dk'_j . Meanwhile, dominated convergence as in Lemma 3.1 gives convergence of the escort densities in L^1 , so their distribution functions $F_{\alpha,j}$ converge uniformly to F_α .

Now choose a continuous function $\chi : I \rightarrow [0, 1]$ that is zero near both endpoints, and insert the factor $\chi(s)\chi(t)$ in (4.1). All these truncated integrals are confined to a common compact square. For a product of continuous test functions $a(s)b(t)$, convergence follows by multiplying the two one-dimensional limits. Uniform approximation by finite sums of these products, together with the mass bounds just proved, gives convergence for any continuous kernel on that square. Since the kernels formed from $F_{\alpha,j}$ also converge uniformly, the truncated integrals converge; and each untruncated integral dominates its truncation. Taking a lower limit in j and then a sequence of such cutoffs increasing to 1 on I gives

$$Q_\alpha(K, u) \leq \liminf_{j \rightarrow \infty} Q_\alpha(K_j, u)$$

by monotone convergence. Varying the projection direction is equivalent to rotating the body while keeping the direction fixed. Since rotations vary continuously in the Hausdorff metric, this inequality also proves lower semicontinuity of $\theta \mapsto Q_\alpha(\theta)$, hence measurability.

The measure inequality. Choose the K_j to be polygons and denote the corresponding quantities by $N_{\alpha,j}$, $r_{\alpha,j}$, and $Q_{\alpha,j}$. Lemma 3.1 gives pointwise convergence $r_{\alpha,j} \rightarrow r_\alpha$. These profiles have a common bound: for the coordinate unit vectors $\mathbf{e}_1, \mathbf{e}_2$, the norm property from Theorem 3.4 gives

$$r_{\alpha,j}(\theta) \leq |\cos \theta| N_{\alpha,j}(\mathbf{e}_1) + |\sin \theta| N_{\alpha,j}(\mathbf{e}_2) \leq N_{\alpha,j}(\mathbf{e}_1) + N_{\alpha,j}(\mathbf{e}_2),$$

and both terms on the right converge by Lemma 3.1. The edge-normal directions of all the K_j form a countable set, which does not affect the following integrals. For every smooth nonnegative test function φ on the circle, the lower-limit inequality above, Fatou's lemma, and Proposition 3.3 give

$$\begin{aligned} \alpha \int \varphi r_\alpha Q_\alpha d\theta &\leq \liminf_{j \rightarrow \infty} \alpha \int \varphi r_{\alpha,j} Q_{\alpha,j} d\theta \\ &\leq \lim_{j \rightarrow \infty} \int r_{\alpha,j} (\varphi + \varphi'') d\theta \\ &= \int r_\alpha (\varphi + \varphi'') d\theta, \end{aligned}$$

where the final equality follows by dominated convergence using the common bound. Theorem 3.4 and the support-function criterion make $r_\alpha + r''_\alpha$ a finite nonnegative measure. Taking $\varphi = 1$ above and using $\min r_\alpha > 0$ proves that Q_α is integrable. The displayed inequality therefore gives (4.2).

The equality case. The kernel in (4.1) is continuous and strictly positive on I^2 , because the escort density is positive on I . If both slope measures are nonzero, then each has positive mass on some compact subinterval, and the kernel has a positive minimum on the product of those intervals. Thus $Q_\alpha > 0$. Conversely, a zero slope measure makes the integral zero. A slope measure vanishes exactly when the corresponding slope is constant, or equivalently when the boundary graph is affine. $\square$

The geometric consequence of the equality case is isolated in the following lemma.

Lemma 4.3. *Let K be a planar convex body. Suppose that, in every projection direction, at least one of the upper and lower boundary functions is affine on the interior of the projected support. Then K is a triangle.*

Proof. An affine boundary graph contains a nondegenerate boundary segment. Let E be the entire intersection of K with the supporting line containing such a segment, and choose the projection direction perpendicular to E .

We first show that both boundary graphs in this direction are affine. Suppose instead that one is nonaffine. By concavity or convexity, it satisfies a strict chord inequality at three abscissae in the interior of the projected support. The local uniform convergence of boundary graphs under rotations, established in the proof of Proposition 4.2, preserves this strict inequality for all sufficiently small rotations of the projection direction. Thus this graph remains nonaffine.

Choose the sign of rotation so that E belongs to the other boundary graph. If $L_E > 0$ is the length of E , its projected length after a rotation through δ is $L_E |\sin \delta|$, whereas the projected width of K tends to a positive number as $\delta \rightarrow 0$. For sufficiently small nonzero δ , the projection of E is therefore a proper subinterval of the projected support. If the graph containing E were affine, its closure would be a collinear boundary segment strictly containing E , contrary to the choice of E as the entire supporting face. Both boundary graphs would then be nonaffine, contradicting the hypothesis.

Hence both boundary graphs are affine in the direction perpendicular to E . Let F be the opposite supporting face. By convexity and compactness, the upper and lower boundary functions extend continuously to the endpoints of the projected support, where their values are the endpoints of the supporting faces. The two affine graphs therefore join the endpoints of E to those of F , and

$$K = \text{conv}(E \cup F).$$

If F is a point, then K is a triangle.

Otherwise, K must be a quadrilateral with parallel supporting sides E and F . A sufficiently small nonzero rotation of the projection direction places these sides on opposite boundary graphs, each projecting onto a proper subinterval of the projected support. Since each side is an entire supporting face, the preceding argument shows that neither graph can be affine; this again contradicts the hypothesis. $\square$

We now prove Theorem 1.2. For a convex body K , its *difference body* is $K - K = \{x - y : x, y \in K\}$, and its *width function* is $w_K(v) = h_K(v) + h_K(-v) = h_{K-K}(v)$. The polar body of $K - K$ is therefore $(K - K)^\circ = \{v : w_K(v) \leq 1\}$.

Proof of Theorem 1.2. First suppose that $K = K_\Delta$ is a triangle. Fix $v \neq 0$, let $y_1 \leq y_2 \leq y_3$ be the values of $\langle v, \cdot \rangle$ at its vertices, and put $w = y_3 - y_1 = w_{K_\Delta}(v) > 0$. The density f of $\langle v, X \rangle$ is affine on each nondegenerate interval among $[y_1, y_2]$ and $[y_2, y_3]$, with height $2/w$ at y_2 and value zero at the other endpoint of each such interval. Parametrizing each piece by $q \in [0, 1]$ gives

$$\int f^\alpha = w \int_0^1 \left(\frac{2q}{w}\right)^\alpha dq = \frac{2^\alpha w^{1-\alpha}}{\alpha + 1}, \quad - \int f \log f = -w \int_0^1 \frac{2q}{w} \log \frac{2q}{w} dq = \log \frac{w}{2} + \frac{1}{2}.$$

Consequently, we have

$$(4.3) \quad \begin{aligned} N_\alpha &= c_\alpha w_{K_\Delta}, & \mathcal{B}_\alpha &= c_\alpha^{-1} (K_\Delta - K_\Delta)^\circ, \\ c_\alpha &= \begin{cases} (2^\alpha / (\alpha + 1))^{1/(1-\alpha)}, & \alpha \neq 1, \\ \sqrt{e}/2, & \alpha = 1. \end{cases} \end{aligned}$$

The difference body $K_\Delta - K_\Delta$ and its polar body are centrally symmetric hexagons, proving the “if” direction.

For the “only if” direction, suppose that $\mathcal{B}_\alpha$ is a polygon. Its polar body is also a polygon, so the surface area measure $r_\alpha + r_\alpha''$ is supported on finitely many directions. By Proposition 4.2, this measure dominates the nonnegative absolutely continuous measure $\alpha r_\alpha Q_\alpha d\theta$, which must therefore vanish. Since $\alpha r_\alpha > 0$, we have $Q_\alpha = 0$ almost everywhere.

Lower semicontinuity gives $Q_\alpha = 0$ everywhere: a positive value would force a positive lower bound on an open interval. The equality statement of Proposition 4.2 therefore shows that, in every direction, at least one boundary graph is affine. Lemma 4.3 then implies that K must be a triangle. $\square$

As a check on Theorem 3.2, consider the normal $\mathbf{u}_{\theta_j}$ of an edge of K_Δ of length L . The opposite face is a vertex, the chord length decreases linearly from L to 0 across the width $w = 2|K_\Delta|/L$, and $\int \ell^\alpha = wL^\alpha/(\alpha + 1)$, so $\Delta_j = L/w$. The side of $\mathcal{B}_\alpha^\circ$ with outer normal $\mathbf{u}_{\theta_j}$ therefore has length $r_\alpha(\theta_j)\Delta_j = c_\alpha L$, as it must for $\mathcal{B}_\alpha^\circ = c_\alpha(K_\Delta - K_\Delta)$.

Remark 4.4 (Endpoint orders). For $\alpha = 0$, $N_0(\mathbf{v})$ is the length of the support of $\langle \mathbf{v}, \mathbf{X} \rangle$, so $N_0 = w_K = h_{K-K}$ and $\mathcal{B}_0 = (K - K)^\circ$. For $\alpha = \infty$, write ℓ_u for the chord-length function perpendicular to a unit vector $\mathbf{u}$. Then $h_\infty(\langle \mathbf{u}, \mathbf{X} \rangle) = \log |K| - \log \max_z \ell_u(z)$, and $\max_z \ell_u(z)$ is the length of the longest chord of K orthogonal to $\mathbf{u}$. It equals the radial function of $K - K$ in that chord direction: for $\mathbf{x}, \mathbf{y} \in K$, the relation $\mathbf{x} - \mathbf{y} = t\mathbf{u}^\perp$ means that they lie on a common chord perpendicular to $\mathbf{u}$, at signed distance t . Hence $\mathcal{B}_\infty$ is the image of $|K|^{-1}(K - K)$ under a rotation by $\pi/2$. Both endpoint unit balls are polygons when K is, which explains why Theorem 1.2 excludes the endpoint orders.

5. EXAMPLES AND REMARKS

5.1. The square. The square checks both the covariance density and the edge-normal atoms. Unlike the triangle, it gives a unit ball with positively curved arcs meeting at corners at every finite positive order.

Let $K = [-1, 1]^2$, $0 < \alpha < \infty$, $0 < \theta < \pi/4$, and $s = \tan \theta$. The marginal density of $\langle \mathbf{u}_\theta, \mathbf{X} \rangle$ equals $1/(2 \cos \theta)$ on the middle interval $[-\cos \theta + \sin \theta, \cos \theta - \sin \theta]$. It is affine on the two outer intervals, each of length $2 \sin \theta$, and vanishes at the support endpoints. Integrating these three pieces gives

$$H_\alpha(\theta) = \log(2 \cos \theta) + \frac{1}{1 - \alpha} \log\left(1 - \frac{\alpha - 1}{\alpha + 1} s\right) \quad (\alpha \neq 1), \quad H_1(\theta) = \log(2 \cos \theta) + \frac{s}{2}.$$

Each boundary chain has one internal vertex. In the original coordinates these are $(-1, 1)$ on the upper chain and $(1, -1)$ on the lower chain; their projections are the left and right endpoints of the middle interval, respectively. On that interval both boundary slopes equal $-s$; on either outer interval one equals $1/s$ and the other equals $-s$. Thus both slope jumps have magnitude $s + 1/s$. Each outer interval has escort mass $s/((\alpha + 1)(1 - s) + 2s)$, so (2.7) gives

$$\frac{r_\alpha + r_\alpha''}{r_\alpha}(\theta) = \frac{\alpha \sec^4 \theta}{((\alpha + 1) - (\alpha - 1) \tan \theta)^2}.$$

At $\alpha = 1$ this is $\frac{1}{4} \sec^4 \theta$; differentiating H_α directly gives the same result. At $\theta = 0$, both supporting faces have length 2 and $\int \ell^\alpha = 2^{\alpha+1}$, so Theorem 3.2 gives $[H'_\alpha]_0 = 2/(\alpha + 1)$, in agreement with $H'_\alpha(0 \pm) = \pm 1/(\alpha + 1)$. Since $r_\alpha(0) = 2$, the atom of $r_\alpha + r_\alpha''$ at 0 has mass $4/(\alpha + 1)$. Symmetry gives the other three coordinate directions and the intervening density. Thus $\mathcal{B}_\alpha$ has positive curvature away from its four corners, while $\mathcal{B}_\alpha^\circ$ has four sides of length $4/(\alpha + 1)$ joined by curved arcs.

At Shannon order, the corresponding boundary points in Figure 2 have the explicit parametrizations

$$\mathbf{b}(\theta) = \frac{1}{2}e^{-s/2}(1, s), \quad \gamma(\theta) = e^{s/2}(2 - s, 1), \quad s = \tan \theta \in [0, 1].$$

Reflection in the coordinate axes and the diagonal gives the other arcs; the polar unit ball also has the four sides $x = \pm 2$, $|y| \leq 1$, and $y = \pm 2$, $|x| \leq 1$.

Figure 2 compares the triangle and square at Shannon order. In both rows, the marginal entropies determine the support distances $r_1(\theta)$ of the polar unit ball and the radial lengths $1/r_1(\theta)$ of the unit ball:

$$\mathcal{B}_1^\circ = \bigcap_{0 \leq \theta < 2\pi} \{\mathbf{y} : \langle \mathbf{y}, \mathbf{u}_\theta \rangle \leq r_1(\theta)\}, \quad \mathcal{B}_1 = \bigcup_{0 \leq \theta < 2\pi} [0, \mathbf{u}_\theta/r_1(\theta)].$$

The square also shows why Theorem 1.2 excludes the endpoint orders: its curved arcs at finite positive orders give way to polygonal unit balls at $\alpha = 0$ and $\alpha = \infty$. For the polar unit balls, the angular measures describe this change explicitly. By Remark 4.4,

$$\mathcal{B}_0^\circ = [-2, 2]^2, \quad \mathcal{B}_\infty^\circ = \text{conv}\{(\pm 2, 0), (0, \pm 2)\}.$$

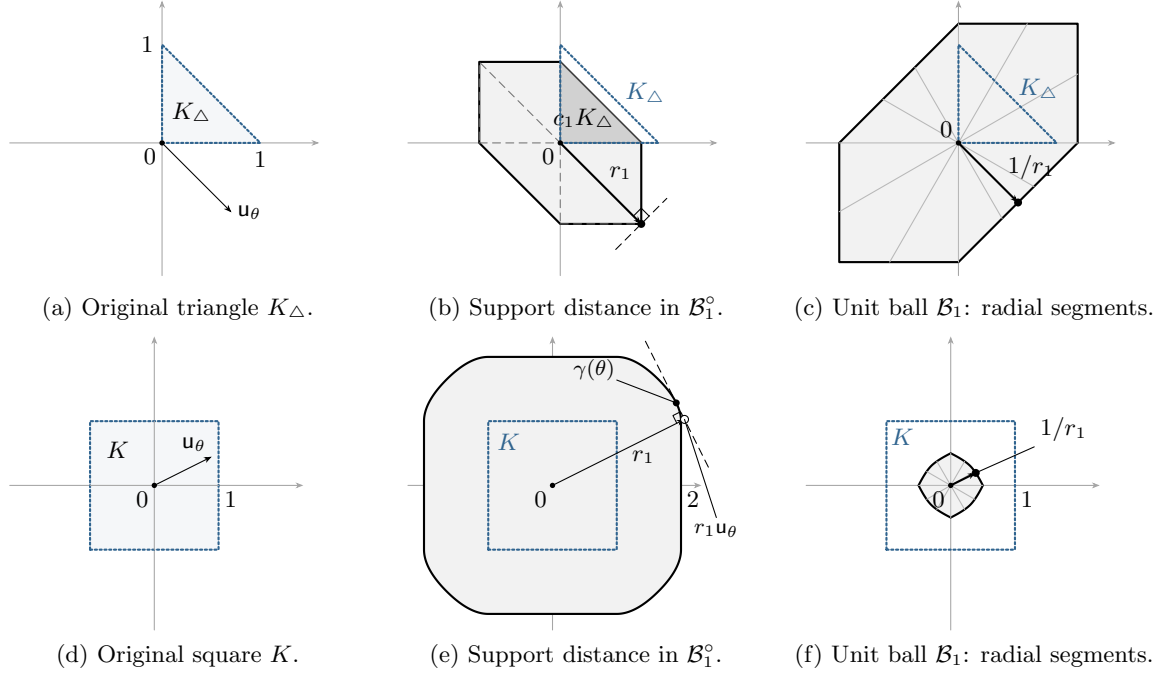


FIGURE 2. The Shannon polar unit ball $\mathcal{B}_1^\circ$ and unit ball $\mathcal{B}_1$, starting from a triangle (top) and a square (bottom). A support distance $r_1(\theta)$ in the middle column gives a radial length $1/r_1(\theta)$ in the final column.

Each row uses the same coordinate scale and aligned origins. Blue dotted outlines retain the original body at the same size and position; polarity relates the middle and final columns. *Top row.* Take $K_\Delta = \text{conv}\{0, \mathbf{e}_1, \mathbf{e}_2\}$, $\mathbf{u}_\theta = (1, -1)/\sqrt{2}$, and $c_1 = \sqrt{e}/2$. The shaded triangle $c_1 K_\Delta$ and its translates by $-c_1 \mathbf{e}_1$ and $-c_1 \mathbf{e}_2$ have convex hull $\mathcal{B}_1^\circ = c_1(K_\Delta - K_\Delta)$. Its support distance in the marked direction is $r_1(\theta) = c_1 \sqrt{2}$. *Bottom row.* Take $K = [-1, 1]^2$ and $\mathbf{u}_\theta = (2, 1)/\sqrt{5}$. In (e), the dashed line supports $\mathcal{B}_1^\circ$ at the filled point $\gamma(\theta)$. The open circle is the perpendicular foot $r_1(\theta)\mathbf{u}_\theta$; it lies outside the polar unit ball and differs from the contact point. In both rows, the final column shows radial segments of length $1/r_1(\theta)$, with the segment in the marked direction emphasized.

As $\alpha \downarrow 0$, the absolutely continuous part of $r_\alpha + r_\alpha''$ loses all its mass, and the coordinate-direction atoms tend to mass 4, the side length of $\mathcal{B}_0^\circ$. As $\alpha \rightarrow \infty$, those atoms vanish, while the absolutely continuous part concentrates at the diagonal directions into atoms of mass $2\sqrt{2}$, the side length of $\mathcal{B}_\infty^\circ$. These are weak limits of measures: monotonicity of Rényi entropy and Dini's theorem give uniform convergence of r_α to each continuous endpoint support function, hence distributional convergence of $r_\alpha + r_\alpha''$. Their total masses $\int_0^{2\pi} r_\alpha(\theta) d\theta$ stay bounded, so convergence also holds against every continuous test function.

5.2. Regularity. For every fixed $0 < \alpha < \infty$, the C^2 regularity in Theorem 1.1 is optimal as a general statement for polygons: C^3 regularity can fail away from edge normals. On each open arc of directions on which the left-to-right order of the projected vertices is fixed, H_α is real-analytic: the entropy of a piecewise linear density is an explicit elementary function of its breakpoints and of its values there, and in the frame of (2.1) these depend affinely and rationally on t . Interior breakpoint heights are positive, while the two extreme heights remain 0, so noninteger powers cause no loss of analyticity. Away from edge normals, the order changes when a vertex of the upper chain and a vertex of the lower chain have the same projection, and there H_α''' jumps in general. For the square, symmetry gives $H'_\alpha(\pi/4) = 0$. Since $(r_\alpha + r_\alpha'')/r_\alpha = 1 + H_\alpha'' + (H_\alpha')^2$, differentiating the formula in Section 5.1 from the left and reflecting across $\pi/4$ gives

$$H_\alpha'''(\pi/4 \mp) = \pm 2\alpha(\alpha + 1) \quad (0 < \alpha < \infty).$$

5.3. Beyond polygons. For the unit disk, H_α is constant, so $(r_\alpha + r''_\alpha)/r_\alpha = 1$. Here $k = -g$ and $g'(z) = -z/\sqrt{1-z^2}$ is unbounded, so Theorem 1.1 does not apply. Nevertheless the escort density is proportional to $(1-z^2)^{\alpha/2}$, $\mathbb{E}g'(Z_\alpha) = 0$, and

$$-\alpha \operatorname{Cov}(g'(Z_\alpha), k'(Z_\alpha)) = \alpha \mathbb{E}g'(Z_\alpha)^2 = \alpha \frac{B(3/2, \alpha/2)}{B(1/2, 1 + \alpha/2)} = 1,$$

where B is the beta function; thus the identity of Theorem 1.1 persists. Some care is needed in general. For $K = \{|x_1|^q + |x_2|^q \leq 1\}$ with $q > 2$ and $\theta = 0$, the boundary point in direction u_0 is flat of order q . Near $z = 1$ one has $\ell \asymp (1-z)^{1/q}$ and $g'^2 \asymp (1-z)^{2/q-2}$, so $\mathbb{E}g'(Z_\alpha)^2 = \infty$ when $q \geq \alpha + 2$, with logarithmic divergence at equality. Thus the formal covariance can be infinite in an individual direction, and a pointwise extension of the covariance identity requires appropriate integrability hypotheses.

Proposition 4.2 nevertheless controls the general-body integral (4.1): it is integrable in the angular variable and gives a lower bound for $r_\alpha + r''_\alpha$. This suffices for Theorem 1.2, but does not identify the remaining nonnegative measure. Whether that remainder consists exactly of the edge-normal atoms with the values in Theorem 3.2, with no additional absolutely continuous or singular continuous part, is not settled by this argument. Nor does the lower bound assert C^2 regularity or a positively curved smooth arc for a general body.

5.4. General log-concave densities. Uniformity enters our argument through the conditional distributions: given $Z = z$, a uniform point of K is uniform on the chord $[k(z), g(z)]$. Its endpoints therefore determine both the marginal density $\ell/|K|$ and its variation under the shear. For a nonuniform log-concave density supported on a convex body, the chord endpoints no longer determine the conditional law or the marginal density. Consequently, monotonicity of the boundary slopes alone does not yield the second-variation bound needed for the full Ball–Nayar–Tkocz norm conjecture.

A decomposition into uniform level-set laws would require an additional argument, since exponentiated Rényi entropy is nonlinear in the density.

REFERENCES

- [1] K. Ball, *Logarithmically concave functions and sections of convex sets in $\mathbb{R}^n$* , Studia Math. **88** (1988), no. 1, 69–84.
- [2] K. Ball, P. Nayar, and T. Tkocz, *A reverse entropy power inequality for log-concave random vectors*, Studia Math. **235** (2016), no. 1, 17–30.
- [3] H. Busemann, *A theorem on convex bodies of the Brunn–Minkowski type*, Proc. Natl. Acad. Sci. USA **35** (1949), 27–31.
- [4] R. J. Gardner, *Geometric Tomography*, 2nd ed., Encyclopedia of Mathematics and its Applications, vol. 58, Cambridge University Press, New York, 2006.
- [5] R. J. Gardner and A. A. Giannopoulos, *p-cross-section bodies*, Indiana Univ. Math. J. **48** (1999), no. 2, 593–613.
- [6] R. J. Gardner and G. Zhang, *Affine inequalities and radial mean bodies*, Amer. J. Math. **120** (1998), no. 3, 505–528.
- [7] J. Haddad, *Planar radial mean bodies are convex*, arXiv:2412.01475v2, 2024.
- [8] S. D. Kominers, *A 9/16 exponent for the Ball–Nayar–Tkocz reverse entropy power inequality*, Harvard Business School Working Paper no. 27-026, September 2026.
- [9] D. Langharst, *Convexity of radial mean bodies via an extension of Ball’s bodies*, arXiv:2603.14134v4, 2026.
- [10] E. L. Lehmann, *Some concepts of dependence*, Ann. Math. Statist. **37** (1966), no. 5, 1137–1153.
- [11] J. Li, *Rényi entropy power inequality and a reverse*, Studia Math. **242** (2018), no. 3, 303–319.
- [12] M. Madiman, J. Melbourne, and P. Xu, *Forward and reverse entropy power inequalities in convex geometry*, in *Convexity and Concentration* (E. Carlen, M. Madiman, and E. M. Werner, eds.), IMA Volumes in Mathematics and its Applications, vol. 161, Springer, New York, 2017, pp. 427–485.
- [13] A. Prékopa, *On logarithmic concave measures and functions*, Acta Sci. Math. (Szeged) **34** (1973), 335–343.
- [14] R. Schneider, *Convex Bodies: The Brunn–Minkowski Theory*, 2nd expanded ed., Encyclopedia of Mathematics and its Applications, vol. 151, Cambridge University Press, Cambridge, 2014.