

A Fibonacci Analogue of the Fifteen Theorem

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Abstract

We prove that a positive-definite quadratic form with integer Gram matrix represents every Fibonacci number if and only if it represents the ten values

$$\begin{aligned} &1, 2, 3, 5, 21, 55, 610, 987, 832040, 267914296 \\ &= F_1, F_3, F_4, F_5, F_8, F_{10}, F_{15}, F_{16}, F_{30}, F_{42}. \end{aligned}$$

The result is a sparse-sequence analogue of the Conway–Schneeberger Fifteen Theorem, with $F_{42} = 267914296$ in the role of 15.

1 Introduction

The Conway–Schneeberger Fifteen Theorem [7, 3] states that a positive-definite quadratic form with integral Gram matrix represents every positive integer if and only if it represents

$$\{1, 2, 3, 5, 6, 7, 10, 14, 15\}.$$

We ask the corresponding question for the Fibonacci numbers. Put $F_0 = 0$, $F_1 = 1$, and $F_{n+2} = F_{n+1} + F_n$, and write $\mathcal{F} = \{F_n : n \geq 1\}$. A form representing every element of $\mathcal{F}$ is *Fibonacci-universal*.

For a set $S \subseteq \mathbb{Z}_{>0}$, an *S-criterion set* is a finite subset $D \subseteq S$ such that every positive-definite quadratic form with integral Gram matrix representing D also represents S . An $\mathcal{F}$ -criterion set is called a *Fibonacci criterion set*.

Existence of a finite criterion set is known in general [13], as is uniqueness of the minimal criterion for integer targets [14, Theorem 2.2]. Our result identifies that set explicitly for $\mathcal{F}$.

Theorem 1.1 (Fibonacci criterion). *A positive-definite quadratic form with integral Gram matrix is Fibonacci-universal if and only if it represents every element of*

$$\begin{aligned} C &= \{1, 2, 3, 5, 21, 55, 610, 987, 832040, 267914296\} \\ &= \{F_1, F_3, F_4, F_5, F_8, F_{10}, F_{15}, F_{16}, F_{30}, F_{42}\}. \end{aligned}$$

Moreover, every Fibonacci criterion set contains C . Thus C is the unique minimal Fibonacci criterion set with respect to inclusion.

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The scale differs markedly from the Fifteen Theorem: the largest required value is $F_{42} = 267914296$. Equivalently, a form is Fibonacci-universal if it represents $F_1, \dots, F_{42}$, and the index 42 cannot be reduced. The reason for this large endpoint is visible at the ternary level. The form $x^2 + y^2 + 2z^2$ omits exactly the positive integers of the form $4^a(16b + 14)$, with integers $a, b \geq 0$, and the first Fibonacci number in that family is F_{42} .

The sparsity of the targets is quantitative: as $X \rightarrow \infty$, Binet's formula gives

$$\#(\mathcal{F} \cap [1, X]) = \frac{\log X}{\log \phi} + O(1), \quad \phi = \frac{1 + \sqrt{5}}{2}.$$

Sparsity alone need not make a criterion difficult. Representing 1 gives every square, and representing 1 and 2 gives every power of 2, simply by scaling vectors. The criterion here has a different role: it forces, for example, representation of $F_{13} = 233$, although 233 is not an integer square multiple of any member of C .

The argument

The proof follows Conway's *escalation* method, as developed in Bhargava's proof of the Fifteen Theorem [3]. Begin with the zero lattice and repeatedly adjoin a vector representing the least Fibonacci value still missing, called the *truant*. We find nine ternary lattices in this process. Their truants, together with those at the earlier ranks, are exactly the ten values in C .

This turns the theorem into an extension problem: show that adjoining the truant to any of the nine ternaries produces a Fibonacci-universal lattice. Such a quaternary extension contains an orthogonal sublattice $T \perp \langle q \rangle$, where T is the ternary parent and q is a bounded positive integer. Thus it is enough to choose k so that the residual

$$F_n - qk^2$$

is nonnegative and represented by T . The bound on q is uniform over all extensions of a given parent.

Eight parents are *regular*: they represent every positive integer that is represented locally at every prime. Each has just one local obstruction, described by a valuation and a residue condition. In most cases, a small choice of k moves the residual out of that obstruction, and Fibonacci growth makes it positive. Direct constructions in other cosets treat the extensions for which the orthogonal sublattice alone does not suffice. For the two odd-prime parents, these constructions come from universal quadratic polynomials derived from the parents' own representation criteria. All eight regular branches therefore close without exhaustive quaternary representation searches.

The ninth parent is

$$f = x^2 + 2y^2 + 2yz + 4z^2.$$

Its genus has two classes, represented by f and $h = x^2 + y^2 + 7z^2$. A locally represented integer is therefore represented by f or h , but we need it represented by f . The key observation is that, for a suitable finite set of primes $p \neq 7$, explicit integral matrices B_p satisfy

$$f(B_p v) = p^2 h(v).$$

Consequently, a locally represented residual with a factor p^2 is represented by f : whichever form represents its quotient, multiplying or applying B_p transfers the representation to f .

We arrange this factor by congruences. A finite cover of the Fibonacci residues selects a prime p and a class of k modulo p^2 ; a separate choice modulo 7 makes the residual locally represented.

The Chinese remainder theorem combines the two choices. After removing an even power of 7 from the target, a growth estimate applies for $n \geq 73$. Exact residual certificates handle $48 \leq n \leq 72$, while the parent and its truant handle the earlier indices.

The transfer and congruence argument applies more broadly than to the Fibonacci numbers. In §6.2 we state it for a general two-class genus. For a target sequence, periodic residues can make a proposed congruence cover finite to check, while valuation bounds and growth give an explicit tail threshold after normalization. The transfer and Chinese remainder steps are uniform; the parent forms, valuation estimates, and successful congruence cover must be established for each sequence.

The paper follows these reductions. §2 collects the Fibonacci arithmetic, §3 determines the ternary parents, and §4 constructs and bounds orthogonal sublattices of their children. Sections 5 and 6 close the branches. In §7, an elementary four-square construction shows that none of the ten truants can be omitted from any Fibonacci criterion. §8 discusses other target sequences, and §A describes the companion library and its verification protocol.

Context

For integer targets, Kim–Lee–Oh [14, Theorem 2.2] identify the unique minimal criterion set with the set of all possible truants; their Remark 2.3 credits an earlier unpublished result of Bhargava. Kala–Krásenský–Romeo [10, Theorem 3.3] develop the corresponding theory over totally real number fields. Our minimality argument follows their extension principle [10, Proposition 3.1], using Kim–Lee–Oh’s four-square construction [14, Proposition 2.1] for the explicit parents in the Fibonacci tree. We include the proof for completeness. The contributions here are the identification of the criterion and the unconditional closure of its irregular branch.

Related explicit criteria include the 290-theorem for integer-valued quadratic forms [4] and Rouse’s work on forms representing all odd positive integers [20]. Sparse targets also have precedents: Ju–Kim–Kim–Kim–Oh prove a 67-criterion for diagonal forms representing all primes [9, Theorem 1.1]. Studies of individual Fibonacci numbers represented by binary quadratic forms include [2, 1].

The irregular form f belongs to Kaplansky’s first nontrivial ternary genus [11, 12]. Analytic treatments of this genus [19, 5] use modular forms and bounds for L -functions, following the approach of Ono–Soundararajan [17]. For example, under GRH for the relevant L -functions, Chandee [5, §4] proves that both genus classes represent every locally represented squarefree integer $N \geq 2 \cdot 10^8$. For the Fibonacci extensions, our square transfers and congruence cover supply the needed residuals unconditionally. This uses the two-class genus description, without requiring a classification of all integers omitted by f .

Notation and conventions

By a *quadratic form* (or just *form*) we mean an expression $Q(x) = x^\top Gx$ with G integral, symmetric, and positive definite. In particular, all cross-terms have even coefficients. A lattice is a free abelian group carrying such a form, and its determinant is the determinant of a Gram matrix. For a lattice L , write Q_L for its quadratic form and β_L for its associated bilinear pairing; in a basis with Gram matrix G , these are $Q_L(x) = x^\top Gx$ and $\beta_L(x, y) = x^\top Gy$. We identify forms with their based lattices, and say that a form *represents* n when $Q(x) = n$ for an integral vector x . The notation $\langle a_1, \dots, a_r \rangle$ denotes the diagonal lattice $a_1x_1^2 + \dots + a_rx_r^2$, and $L \perp K$ denotes an orthogonal direct sum.

Two lattices are in the same *genus* if they are isometric over $\mathbb{R}$ and over every $\mathbb{Z}_p$. Write $L_p = L \otimes \mathbb{Z}_p$. A positive integer is *eligible*, or *locally represented*, for the genus if every L_p represents it.

The local–global principle for representation by a genus says that this is equivalent to representation by some lattice in the genus. A form is *regular* if it represents every eligible positive integer, and *irregular* otherwise.

For a prime p and $N > 0$, let $v_p(N)$ denote its p -adic valuation and $u_p(N) = N/p^{v_p(N)}$ its prime-to- p part. We use the same valuation on $\mathbb{Q}_p$, with $v_p(0) = +\infty$. We write T for a ternary parent and M for a quaternary extension. Finally, $E(L) \subseteq \mathbb{Z}$ denotes the integers not represented by L ; it includes all negative integers and excludes 0. This convention lets us say that a residual lies outside $E(L)$ without separately requiring nonnegativity. The *Fibonacci exceptions* of L are $\mathcal{F} \cap E(L)$.

2 Fibonacci arithmetic

The representation criteria below depend on two pieces of information about an integer N : the valuation $v_p(N)$ at one prime, and the residue of its unit part $u_p(N)$. We first record the Fibonacci identities that determine these quantities. We also need a simple growth estimate: it will turn a bound at one index into a bound for every later index.

Lemma 2.1 (Fibonacci growth). *The sequence F_n/n is strictly increasing for $n \geq 2$.*

Proof. The inequality $nF_{n+1} > (n+1)F_n$ is equivalent to $nF_{n-1} > F_n$. It holds directly for $n = 2$; for $n \geq 3$, use $F_n = F_{n-1} + F_{n-2} \leq 2F_{n-1} < nF_{n-1}$. $\square$

For a prime p , its *rank of apparition* $\rho_F(p)$ is the least positive index at which p divides a Fibonacci number. The standard Fibonacci divisibility property gives $p \mid F_n$ if and only if $\rho_F(p) \mid n$.

Lemma 2.2 (Fibonacci valuations). *For every $n \geq 1$,*

$$v_2(F_n) = \begin{cases} 0, & 3 \nmid n, \\ 1, & n \equiv 3 \pmod{6}, \\ v_2(n) + 2, & 6 \mid n. \end{cases}$$

The ranks at 3, 5, and 7 are $\rho_F(3) = 4$, $\rho_F(5) = 5$, and $\rho_F(7) = 8$, and

$$v_3(F_n) = \begin{cases} 1 + v_3(n), & 4 \mid n, \\ 0, & 4 \nmid n, \end{cases} \quad v_5(F_n) = v_5(n),$$

$$v_7(F_n) = \begin{cases} 1 + v_7(n), & 8 \mid n, \\ 0, & 8 \nmid n. \end{cases}$$

Proof. The values $F_4 = 3$, $F_5 = 5$, and $F_8 = 21$, together with the preceding terms, give the three ranks. For an odd prime $p \neq 5$, the Lucas-sequence lifting formula at $m = \rho_F(p)$ is

$$v_p(F_{mk}) = v_p(F_m) + v_p(k).$$

It gives the assertions at 3 and 7. At the discriminant prime 5, the corresponding formula is $v_5(F_n) = v_5(n)$. These formulas and the formula at 2 are given by Lengyel [15, Lemmas 1–2 and the theorem in §3]. The extra term at 2 when $6 \mid n$ can be understood from $F_{2r} = F_r L_r$, where $L_0 = 2$, $L_1 = 1$, and $L_{r+2} = L_{r+1} + L_r$ are the Lucas numbers. For odd m , one has $v_2(F_{3m}) = 1$. The first doubling of the index adds 2, because $v_2(L_r) = 2$ for $r \equiv 3 \pmod{6}$; subsequent doublings add 1, because $v_2(L_r) = 1$ for $r \equiv 0 \pmod{6}$. This gives $v_2(F_n) = v_2(n) + 2$ when $6 \mid n$. $\square$

At each of these fixed primes, the prime-power part of F_n is at most a constant times its index. We will repeatedly use the bounds

$$\begin{aligned} 2^{v_2(F_n)} &\leq 4n, & 3^{v_3(F_n)} &\leq 3n, & 5^{v_5(F_n)} &\leq n, \\ 7^{v_7(F_n)} &\leq \frac{7n}{8} & \text{when } 8 \mid n. \end{aligned} \quad (1)$$

For example, if $8 \mid n$, then $7^{v_7(F_n)} = 7 \cdot 7^{v_7(n/8)} \leq 7n/8$. Dividing F_n by the relevant bound gives a lower bound for $u_p(F_n)$.

Residues at a fixed modulus are equally explicit. The *Pisano period* $\pi(m)$ is the least positive period of the Fibonacci sequence modulo m .

Lemma 2.3 (Pisano periods). *The Fibonacci sequence is periodic modulo every $m \geq 2$. In particular,*

m	3	5	7	8	9	16
$\pi(m)$	8	20	16	12	24	24.

Proof. The state transition $(a, b) \mapsto (b, a + b)$ is invertible modulo m , so its orbit from $(F_0, F_1) = (0, 1)$ is periodic. Iterating until this state first returns gives the displayed periods. See Wall [22] for the systematic theory. $\square$

A fixed Pisano period determines F_n modulo its modulus for every index. To determine $u_p(F_n)$ modulo p^j , however, one must first know $a = v_p(F_n)$ and then compute F_n modulo p^{a+j} . This distinction is useful: finite recurrence calculations determine the first occurrence of a local obstruction, while (1) and Lemma 2.1 control the sizes of the normalized targets thereafter.

The local obstructions for the ternary lattices will be the six families in the next table. The valuation descriptions explain their common structure: an even power of the governing prime may be removed, after which membership is decided by a small residue class.

Lemma 2.4 (First Fibonacci members of the exceptional families). *For each family below, the rightmost column gives its least Fibonacci member. Here $a, b \geq 0$ are integers.*

family	condition on N	first Fibonacci member
$4^a(8b + 7)$	$v_2(N)$ even, $u_2(N) \equiv 7 \pmod{8}$	$F_{10} = 55$
$4^a(16b + 14)$	$v_2(N)$ odd, $u_2(N) \equiv 7 \pmod{8}$	$F_{42} = 267914296$
$4^a(16b + 10)$	$v_2(N)$ odd, $u_2(N) \equiv 5 \pmod{8}$	$F_{30} = 832040$
$9^a(9b + 6)$	$v_3(N)$ odd, $u_3(N) \equiv 2 \pmod{3}$	$F_{16} = 987$
$5 \cdot 25^a(5b + r)$ $r \in \{2, 3\}$	$v_5(N)$ odd, $u_5(N) \equiv 2, 3 \pmod{5}$	$F_{15} = 610$
$7 \cdot 49^a(7b + r)$ $r \in \{3, 5, 6\}$	$v_7(N)$ odd, $u_7(N) \equiv 3, 5, 6 \pmod{7}$	$F_8 = 21.$

Proof. The first two columns are equivalent after factoring out the indicated power of the prime. To find the first Fibonacci member, the valuation formulas reduce the search to a short list of indices; the recurrence then gives the unit residues. We record the calculations.

For the two families with odd 2-adic valuation, the relevant indices through 42 and their residues are

n	3	6	9	15	18	21	24	27	30	33	39	42
$v_2(F_n)$	1	3	1	1	3	1	5	1	3	1	1	3
$u_2(F_n) \bmod 8$	1	1	1	1	3	1	1	1	5	1	1	7.

Thus the first unit residue 5 occurs at F_{30} and the first unit residue 7 at F_{42} . For $4^a(8b + 7)$, the indices below 10 with even 2-adic valuation are 1, 2, 4, 5, 7, 8. Their odd parts are 1, 1, 3, 5, 5, 5 modulo 8, whereas $F_{10} = 55 \equiv 7 \pmod{8}$.

For the 3-adic family, the only indices below 16 with odd positive valuation are 4 and 8, and both unit parts are 1 (mod 3). At 16, one has $F_{16} = 3 \cdot 329$ and $329 \equiv 2 \pmod{3}$. For the 5-adic family, the only possible earlier indices are 5 and 10, again with square unit part 1 (mod 5); the first nonsquare unit part is $F_{15}/5 = 122 \equiv 2 \pmod{5}$. Finally, $7 \mid F_n$ first occurs at $F_8 = 21 = 7 \cdot 3$. All these residue calculations use exact Fibonacci recurrence values, or equivalently the recurrence modulo the corresponding prime powers. $\square$

3 The ternary escalation tree

The small ranks determine which values must occur in the criterion set. Starting from the zero lattice, we adjoin a vector representing the least Fibonacci number the current lattice omits. Reduction of the resulting lattice leaves only nine possibilities at rank 3.

3.1 The nine ternary parents

The least Fibonacci number omitted by a lattice L is its *Fibonacci truant*. If this value is τ , an *immediate escalation* of L is an integral lattice $L + \mathbb{Z}u$ with $Q(u) = \tau$; we call L its *parent*. A Fibonacci-universal lattice has no truant and is not escalated. We classify escalations up to integral isometry, allowing the new generator to be changed by a vector of the parent.

The zero lattice has truant 1 and escalates to $\langle 1 \rangle$. Its truant is 2, and a vector of norm 2 cannot lie in its rational span because 2 is not a rational square. A binary escalation has Gram matrix

$$\begin{pmatrix} 1 & a \\ a & 2 \end{pmatrix}, \quad a = 0, \pm 1.$$

The case $a = 0$ gives $\langle 1, 2 \rangle$; subtracting the norm-1 vector from the new vector when $a = 1$, or adding it when $a = -1$, gives $\langle 1, 1 \rangle$. The binary parents thus have truants

$$\langle 1, 1 \rangle : 3, \quad \langle 1, 2 \rangle : 5.$$

These follow by checking the preceding Fibonacci values.

Both binary truants also lie outside the rational space of their parent. Indeed, a rational representation would give $x^2 + y^2 = 3d^2$ or $x^2 + 2y^2 = 5d^2$ after clearing denominators. Reduction modulo 3 or 5, respectively, forces that prime to divide x, y , and then d . Dividing all three by the prime gives an infinite descent. Thus adjoining either truant really does produce a ternary lattice.

Proposition 3.1 (The nine ternaries). *The immediate escalations of the two binary parents are, up to integral isometry, the seven diagonal lattices*

$$\begin{aligned} &\langle 1, 1, 1 \rangle, \quad \langle 1, 1, 2 \rangle, \quad \langle 1, 1, 3 \rangle, \\ &\langle 1, 2, 2 \rangle, \quad \langle 1, 2, 3 \rangle, \quad \langle 1, 2, 4 \rangle, \quad \langle 1, 2, 5 \rangle, \end{aligned}$$

together with $f = x^2 + 2y^2 + 2yz + 4z^2$ and $g = x^2 + 2y^2 + 2yz + 5z^2$, whose Gram matrices are

$$G_f = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 4 \end{pmatrix}, \quad G_g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 5 \end{pmatrix}.$$

Proof. For $\langle 1, 1 \rangle$, subtracting integral multiples of the two old basis vectors makes the new vector orthogonal to them. Its norm becomes $3 - r$, where r is a sum of two squares with $0 \leq r < 3$. The possibilities $r = 0, 1, 2$ give $\langle 1, 1, 3 \rangle$, $\langle 1, 1, 2 \rangle$, and $\langle 1, 1, 1 \rangle$.

For $\langle 1, 2 \rangle$, first remove the pairing with the norm-1 vector. Subtracting multiples of the norm-2 vector reduces its remaining pairing to $b \in \{0, 1\}$. The reduced form is

$$Q(x, y, z) = x^2 + 2y^2 + 2byz + mz^2.$$

The original vector of norm 5 still has new coordinate $z = 1$. Since $2y^2 + 2by \geq 0$ for integral y and $b \in \{0, 1\}$, its representation gives $m \leq 5$; positive definiteness gives $m \geq 1$.

If $b = 0$, the five values of m give $\langle 1, 1, 2 \rangle$ and the four forms $\langle 1, 2, m \rangle$ with $2 \leq m \leq 5$. If $b = 1$ and $m = 1$, the identity

$$x^2 + 2y^2 + 2yz + z^2 = x^2 + y^2 + (y + z)^2$$

gives $\langle 1, 1, 1 \rangle$. The cases $m = 2$ and $m = 3$ do not represent 5: the values at most 5 of their binary parts are, respectively, $\{0, 2\}$ and $\{0, 2, 3\}$, and none completes a square to 5. Finally, $m = 4$ gives f , with $5 = f(1, 0, 1)$, and $m = 5$ gives g , with $5 = g(0, 0, 1)$. Each of the diagonal possibilities also has a representation of the truant with new coordinate 1, so all listed lattices do occur as escalations.

Their determinants, in the displayed order, are 1, 2, 3, 4, 6, 8, 10, 7, 9. They are therefore pairwise inequivalent. $\square$

3.2 The regular parents

For eight of these forms, the local representation criterion is also a global criterion. Several have the same set of represented integers, so we group them by their obstruction. The first-member calculation in Lemma 2.4 then identifies their truant.

Proposition 3.2 (Regular ternaries). *The eight forms below are regular, with the indicated positive exceptional sets and Fibonacci truant. Here $a, b \geq 0$ are integers.*

T	$E(T) \cap \mathbb{Z}_{>0}$	Fibonacci truant
$\langle 1, 1, 1 \rangle, \langle 1, 2, 2 \rangle, g$	$4^a(8b + 7)$	$F_{10} = 55$
$\langle 1, 1, 2 \rangle, \langle 1, 2, 4 \rangle$	$4^a(16b + 14)$	$F_{42} = 267914296$
$\langle 1, 2, 3 \rangle$	$4^a(16b + 10)$	$F_{30} = 832040$
$\langle 1, 1, 3 \rangle$	$9^a(9b + 6)$	$F_{16} = 987$
$\langle 1, 2, 5 \rangle$	$5 \cdot 25^a(5b + r)$ $r \in \{2, 3\}$	$F_{15} = 610.$

Proof. Ramanujan [18, §3, formulas (3.11)–(3.71)] records the exceptional sets of the seven diagonal forms, without giving the ternary proofs. Their regularity is established in Jones–Pall [8, Table I, p. 190]. Thus these classical criteria supply both necessity and sufficiency. We explain the shared obstruction in the first row and the odd-prime obstructions, which will also govern the extension arguments.

The Gauss–Legendre three-square theorem gives the criterion for $\langle 1, 1, 1 \rangle$. The identity

$$x^2 + 2y^2 + 2z^2 = x^2 + (y + z)^2 + (y - z)^2$$

shows that every value of $\langle 1, 2, 2 \rangle$ is a sum of three squares. Conversely, two coordinates in any three-square representation have the same parity; they can therefore be written as $y + z$ and $y - z$. Thus these two forms represent exactly the same integers.

The same holds for g , since

$$g(x, y, z) = x^2 + (y + 2z)^2 + (y - z)^2.$$

Given a sum of three squares, choose two coordinates A, B that are congruent modulo 3, changing a sign if necessary. This is always possible: either at least two coordinates are divisible by 3, or at least two are units. Then

$$y = (A + 2B)/3, \quad z = (A - B)/3$$

are integers, and $A = y + 2z$, $B = y - z$ give a representation by g . Both displayed embeddings into three squares hold over every $\mathbb{Z}_p$. Hence any positive integer locally represented by $\langle 1, 2, 2 \rangle$ or g is locally represented by three squares and therefore represented integrally by all three forms. This also proves their regularity. The other two 2-adic families are precisely the classical diagonal criteria cited above.

For the odd-prime rows, write the form as $x^2 + \alpha y^2 + pz^2$, with $(\alpha, p) = (1, 3)$ or $(2, 5)$. The binary form $x^2 + \alpha y^2$ is anisotropic modulo p . Consequently, if $p \mid n$ is represented, then $x = pX$, $y = pY$, and

$$n = p(pX^2 + \alpha pY^2 + z^2).$$

At valuation 1, the unit part of n must therefore be a nonzero square modulo p . If $p^2 \mid n$, the same identity forces $p \mid z$; dividing all coordinates by p divides the represented value by p^2 . Repeated division leaves an odd valuation obstructed precisely when the unit part is a nonsquare. The nonsquares are $\{2\}$ modulo 3 and $\{2, 3\}$ modulo 5, giving the last two rows. The classical representation criteria supply sufficiency. The Fibonacci truants now follow from Lemma 2.4. $\square$

3.3 The determinant-7 ternary

The ninth ternary is

$$f(x, y, z) = x^2 + 2y^2 + 2yz + 4z^2, \quad G_f = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 4 \end{pmatrix}.$$

Its genus contains two classes. Thus a local representation guarantees a representation by one of two explicit forms; later we will use integral transformations to pass from the second form back to f .

Proposition 3.3 (Genus and local obstruction). *The genus of f consists of the two integral isometry classes represented by f and $h = \langle 1, 1, 7 \rangle$. A positive integer N is eligible for this genus if and only if*

$$v_7(N) \text{ is even} \quad \text{or} \quad u_7(N) \equiv 1, 2, 4 \pmod{7}.$$

In particular, every such N is represented by f or h .

Proof. The two-class genus description is due to Kaplansky [11]; Kelley [12, pp. 289–290] records the pair and its eligibility criterion. We verify the genus description directly before computing the local obstruction.

Every positive-definite integral ternary lattice of determinant 7 has a Minkowski-reduced Gram matrix

$$G = \begin{pmatrix} a & r & s \\ r & b & t \\ s & t & c \end{pmatrix}, \quad 1 \leq a \leq b \leq c, \quad 2|r|, 2|s| \leq a, \quad 2|t| \leq b.$$

The classical ternary reduction bound gives $abc \leq 2 \det G = 14$; see Mahler [16, p. 193, (3)]. Hence $a \in \{1, 2\}$. If $a = 1$, then $r = s = 0$ and $bc - t^2 = 7$. Since $b \leq c$ and $|t| \leq b/2$, we have $3b^2 \leq 28$, so $b \leq 3$. The only possibilities are $(b, c, t) = (1, 7, 0)$ and $(2, 4, \pm 1)$, giving h and f .

If $a = 2$, then $b = 2$ and $c \in \{2, 3\}$, with $|r|, |s|, |t| \leq 1$. The case $c = 2$ has even determinant. For $c = 3$, the determinant equation gives

$$r^2 = 1, \quad s^2 + t^2 - rst = 1.$$

If one of s, t vanishes, changing signs and interchanging the first two basis vectors gives

$$G_k = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & 0 \\ -1 & 0 & 3 \end{pmatrix}.$$

Otherwise $rst = 1$; changing signs makes $r = s = t = 1$, and the basis $(e_1, -e_2, e_3 - e_1)$ again gives G_k . Thus every class of determinant 7 is represented by f , h , or k .

Put $G_h = \text{diag}(1, 1, 7)$. The matrices

$$U_2 = \begin{pmatrix} 0 & -2 & 0 \\ 0 & 0 & 4 \\ -1 & 0 & -1 \end{pmatrix}, \quad U_3 = \begin{pmatrix} -1 & -1 & 7 \\ -1 & 2 & 1 \\ -1 & -1 & -2 \end{pmatrix}$$

satisfy $U_i^T G_f U_i = i^2 G_h$ and $\det U_i = i^3$. Thus $U_2/2$ gives an isometry over every $\mathbb{Z}_p$ with p odd, and $U_3/3$ gives one over $\mathbb{Z}_2$; both also give real isometries. So f and h lie in the same genus. Modulo 7, the nonsingular rank-2 quotients of G_k and G_f have determinants 3 and 2, respectively. These lie in different square classes, excluding k from this genus. Finally, f represents 3, whereas h does not, so the two classes are distinct.

At 7, reduction gives

$$f(x, y, z) \equiv x^2 + 2(y + 4z)^2 \pmod{7}.$$

Every nonzero residue occurs, since both the square residues and twice the square residues are $\{0, 1, 2, 4\}$, whose sum is all of $\mathbb{F}_7$. A nonzero value has a nonsingular representing vector, so Hensel's lemma represents every 7-adic unit.

On the other hand, -2 is a nonsquare modulo 7. Thus $7 \mid f(x, y, z)$ forces $x = 7X$ and $y = -4z + 7Y$, giving

$$f(x, y, z) = 7(7X^2 + 14Y^2 - 14Yz + 4z^2).$$

If the value has valuation 1, its unit part is therefore a nonzero square modulo 7. Conversely, for a square unit $u \in \mathbb{Z}_7^\times$, Hensel's lemma gives $s \in \mathbb{Z}_7^\times$ with $4s^2 = u$, and $f(0, -4s, s) = 28s^2 = 7u$. If the value is divisible by 49, the same displayed identity forces $7 \mid z$, and hence $7 \mid x, y, z$. Dividing the coordinates by 7 then divides the value by 49; scaling gives the converse. This reduces every valuation to 0 or 1 and proves the criterion at 7.

For odd $p \neq 7$, the lattice is unimodular of rank 3. Its reduction modulo p represents every nonzero residue and has a nonzero isotropic vector, so Hensel lifting makes it $\mathbb{Z}_p$ -universal. At 2, let $A \in \mathbb{Z}_2$ and choose $e \in \{0, 1\}$ with $e \equiv A \pmod{2}$. The polynomial

$$2z^2 + z + 1 - \frac{A - e^2}{2}$$

has a simple root modulo 2, and hence a root $z \in \mathbb{Z}_2$. For this root, $f(e, 1, z) = A$. Thus there is no local obstruction away from 7. Finally, representation by every localization is equivalent to representation by some class in the genus, proving the last assertion. $\square$

In particular f is not regular: the value $7 = h(0, 0, 1)$ is eligible but is not represented by f . Indeed, $2y^2 + 2yz + 4z^2 = 2(y + z/2)^2 + 7z^2/2$ takes only the values 0, 2, 4 below 8, and none completes a square to 7. The argument below uses the two-class genus directly. For the initial Fibonacci indices, explicit witnesses suffice.

Certificate 3.4 (Fibonacci interval for f). *The form f represents F_n for every $9 \leq n \leq 47$.*

Proof. §A lists an integral triple (x, y, z) for each of these thirty-nine indices. Substitution gives $f(x, y, z) = F_n$ in every case; `verify_certificates.py` reproduces these exact checks. $\square$

Proposition 3.5 (First Fibonacci exceptions). *The Fibonacci truant of f is $F_8 = 21$, and its next Fibonacci exception is $F_{48} = 4807526976$.*

Proof. The representations

N	1	2	3	5	8	13
(x, y, z)	(1, 0, 0)	(0, 1, 0)	(1, 1, 0)	(1, 0, 1)	(0, 2, 0)	(3, 0, 1)

cover $F_1, \dots, F_7$, while $21 = 7 \cdot 3$ is locally obstructed. The certificate covers $F_9, \dots, F_{47}$. Finally, $F_{48} \equiv 21 \pmod{49}$, so it has valuation 1 at 7 and nonsquare unit part 3 $\pmod{7}$; Proposition 3.3 excludes it. $\square$

The truants encountered in the tree are exactly the elements of C . Table 1 collects one parent for each value. These same parents will provide the omission examples in the minimality proof.

Table 1: The criterion values and parents having them as truants.

Truant	Parent
1	zero lattice
2	$\langle 1 \rangle$
3	$\langle 1, 1 \rangle$
5	$\langle 1, 2 \rangle$
21	f
55	$\langle 1, 1, 1 \rangle$
610	$\langle 1, 2, 5 \rangle$
987	$\langle 1, 1, 3 \rangle$
832040	$\langle 1, 2, 3 \rangle$
267914296	$\langle 1, 1, 2 \rangle$

3.4 Each truant raises the rank

For the unary and binary parents we already checked that adjoining a vector of truant norm raises the rank. The local criteria give the same conclusion for all nine ternaries. This ensures that an escalation inside an ambient lattice really reaches rank 4, rather than merely enlarging a sublattice of the same rational space.

Lemma 3.6 (No same-rank escalation). *For every parent of rank at most 3 in the escalation tree, the truant is not represented by its rational quadratic space. Hence any vector of truant norm in an ambient lattice is linearly independent of the parent.*

Proof. The zero, unary, and binary cases were established above. For a ternary parent, its truant lies in the local obstruction described in Propositions 3.2 and 3.3. Each obstruction is unchanged by multiplication by a nonzero integer square: its valuation changes by an even integer, and its unit part changes by a square. At 2, an odd square is 1 (mod 8); at the odd governing primes, square multiplication preserves whether a unit is a square or nonsquare. Thus every nonzero integer-square multiple of the truant is still locally obstructed. Clearing denominators in a rational representation would produce an integral representation of one of these multiples, a contradiction. $\square$

4 Children of the ternary parents

We now describe the quaternary lattices whose universality will close the tree. Each contains an orthogonal sublattice $T \perp \langle q \rangle$, where T is its ternary parent and q is a bounded positive integer. Thus we can represent a target N by finding k for which the parent represents $N - qk^2$. We first construct this sublattice and bound q , then record the sharper bounds needed at the odd governing primes.

4.1 Children and their orthogonal sublattices

Let T have basis b_1, b_2, b_3 , Gram matrix A , and Fibonacci truant τ . A quaternary extension candidate is a positive-definite integral lattice

$$M = T + \mathbb{Z}v, \quad \text{Gram}(M) = \begin{pmatrix} A & c \\ c^\top & m \end{pmatrix}, \quad c_i = \beta_M(v, b_i), \quad m = Q_M(v).$$

Replacing v by $v + x$ for $x \in T$ changes c to $c + Ax$. Fix representatives for $\mathbb{Z}^3/A\mathbb{Z}^3$; a candidate is *reduced* when c belongs to this set. A reduced candidate representing τ is called a *child* of T . Every immediate escalation reduces to a child, although its reduced generator need no longer have norm τ . For a general child, a vector w representing τ need not have new coordinate ± 1 , so the immediate escalation $T + \mathbb{Z}w$ may be a proper sublattice of the child. Our closure results apply to this larger class of extensions, which includes every immediate escalation.

The Schur complement

$$\delta = m - c^\top A^{-1}c$$

is the norm of the component of v perpendicular to $T \otimes \mathbb{Q}$; positive definiteness is equivalent to $\delta > 0$. Let d be the least positive integer for which $dA^{-1}c \in \mathbb{Z}^3$, equivalently the order of $A^{-1}c$ in $\mathbb{Q}^3/\mathbb{Z}^3$, and put $q = d^2\delta$.

Lemma 4.1 (The orthogonal sublattice). *The integer q is positive, and M contains $T \perp \langle q \rangle$ with index d . Consequently M represents N whenever, for some integer $k \geq 0$, the residual $N - qk^2$ is nonnegative and represented by T . Equivalently, the condition is $N - qk^2 \notin E(T)$.*

Proof. Clearing the denominators of the perpendicular component gives

$$w_\perp = dv - \sum_{i=1}^3 (dA^{-1}c)_i b_i \in M.$$

This vector is orthogonal to T and has norm $d^2\delta = q$. Since M has integral Gram matrix, $q \in \mathbb{Z}_{>0}$. Moreover, $T + \mathbb{Z}w_\perp$ has index d in $T + \mathbb{Z}v$, as is seen from the coefficient of v . This proves the containment and index. If $Q_T(x) = N - qk^2$, then $x + kw_\perp$ represents N in M . $\square$

We call this sufficient condition the *trivial-coset test*: it uses the class of 0 in $M/(T \perp \mathbb{Z}w_\perp) \cong \mathbb{Z}/d\mathbb{Z}$. The other classes may supply representations even when the test fails. When $p \nmid d$, the inclusion becomes an equality after tensoring with $\mathbb{Z}_p$; in particular,

$$M_p \cong T_p \perp \langle q \rangle.$$

Representing the truant bounds the perpendicular norm. Indeed, if $Q_M(x + kv) = \tau$, then $k \neq 0$, since T omits τ . Completing the square gives

$$\tau = (x + kA^{-1}c)^\top A(x + kA^{-1}c) + k^2\delta \geq \delta.$$

Thus every child satisfies

$$\delta \leq \tau. \quad (2)$$

The adjugate formula gives $d \mid \det A$, so also

$$q \leq \tau(\det A)^2. \quad (3)$$

For a fixed reduced cross-vector c , the first bound gives the finite range $m \leq \lfloor \tau + c^\top A^{-1}c \rfloor$. For a diagonal parent $A = \text{diag}(a_1, a_2, a_3)$, the quotient $\mathbb{Z}^3/A\mathbb{Z}^3$ is the product of the coordinate quotients $\mathbb{Z}/a_i\mathbb{Z}$. For f and g , let e_i denote the standard coordinate vectors. In the respective quotients, the matrix columns give

	$[e_1]$	$[e_2]$	relation on $[e_3]$
f	0	$-4[e_3]$	$7[e_3] = 0$
g	0	$-5[e_3]$	$9[e_3] = 0$.

Their determinants show that $[e_3]$ has order 7 and 9, respectively. These descriptions justify the complete sets of reduced cross-vectors used below.

The following table gives the coefficient ranges needed explicitly in the closure proofs. Each range is also subject to $\delta > 0$; here $c_1 = 0$ and every unlisted c_i is 0.

parent	$\det A$	representatives for c	m -range
$\langle 1, 2, 2 \rangle$	4	$c_2, c_3 \in \{0, 1\}$	$m \leq 56$
$\langle 1, 1, 3 \rangle$	3	$c_3 \in \{0, 1, 2\}$	$m \leq \lfloor 987 + c_3^2/3 \rfloor$
$\langle 1, 2, 5 \rangle$	10	$c_2 \in \{0, 1\}, c_3 \in \{-2, \dots, 2\}$	$m \leq 611$
f	7	$c_3 \in \{0, \dots, 6\}$	$m \leq 31$

The remaining reductions are introduced where they are used in the 2-adic argument.

Since $T \subseteq M$, a child already represents every value represented by its parent. To prove it Fibonacci-universal, it therefore suffices to represent the parent's Fibonacci exceptions, $\mathcal{F} \cap E(T)$. The truant itself is represented by the definition of a child.

4.2 Bounds at the odd governing primes

For the parents governed by 3, 5, and 7, representing the truant also limits the valuation of q . The following bounds apply throughout both the regular and irregular branch arguments.

Lemma 4.2 (Valuation and size bounds for children). *For a child of $\langle 1, 1, 3 \rangle$, $\langle 1, 2, 5 \rangle$, or f , let p be 3, 5, or 7, respectively. Then $v_p(q) \leq 1$. For a child of f , we also have $u_7(q) \leq 147$; in fact, $q \leq 21$ when $7 \nmid q$.*

Proof. Use the reduced cross-vectors in the table above. First suppose $c_3 \neq 0$. For $\langle 1, 1, 3 \rangle$ we have $d = 3$ and

$$q = 9m - 3c_3^2,$$

so $v_3(q) = 1$. For $\langle 1, 2, 5 \rangle$, we have $d \in \{5, 10\}$ and

$$q = d^2m - \frac{d^2c_2^2}{2} - \frac{d^2c_3^2}{5}.$$

The first two terms are divisible by 25, and the last has valuation 1 at 5, so $v_5(q) = 1$. For f , we have $d = 7$ and

$$q = 49m - 14c_3^2,$$

which likewise has valuation 1 at 7.

If $c_3 = 0$, then $p \nmid d$ in each case. Hence $M_p \cong T_p \perp \langle q \rangle$. Were $p^2 \mid q$, subtracting qk^2 could not change the truant's residue modulo p^2 :

$$987 \equiv 6 \pmod{9}, \quad 610 \equiv 10 \pmod{25}, \quad 21 \equiv 21 \pmod{49}.$$

Each residue lies in its parent's local obstruction, so the candidate could not represent the truant even over $\mathbb{Z}_p$. This proves $v_p(q) \leq 1$.

For f with $7 \mid q$, the bound $q \leq \tau(\det T)^2 = 21 \cdot 7^2$ gives $u_7(q) = q/7 \leq 147$. If $7 \nmid q$, the preceding cases force $c_3 = 0$, hence $d = 1$; then $u_7(q) = q = \delta \leq 21$. $\square$

5 Closing the regular branches

For a regular parent, every nonnegative residual outside its positive exceptional family is represented. We use this fact to close the six 2-adic parents, then the parents governed by 3 and 5.

5.1 The 2-adic parents

The six parents in this branch have three exceptional families:

$$\mathcal{E}_{\varepsilon, r} = \{N \in \mathbb{Z}_{>0} : v_2(N) \equiv \varepsilon \pmod{2}, u_2(N) \equiv r \pmod{8}\}.$$

Their parameters are

T	(ε, r)	exceptional family
$\langle 1, 1, 1 \rangle, \langle 1, 2, 2 \rangle, g$	$(0, 7)$	$4^j(8b + 7)$
$\langle 1, 1, 2 \rangle, \langle 1, 2, 4 \rangle$	$(1, 7)$	$4^j(16b + 14)$
$\langle 1, 2, 3 \rangle$	$(1, 5)$	$4^j(16b + 10)$.

Subtracting qk^2 can leave such a family in either of two ways: it can change the parity of the valuation, or change the odd part modulo 8. The next lemma gives explicit choices of k , and identifies when this orthogonal subtraction cannot work.

Lemma 5.1 (2-adic subtraction). *Let $N = 2^a u \in \mathcal{E}_{\varepsilon, r}$ and $q = 2^s \omega \in \mathbb{Z}_{>0}$, where $a, s \geq 0$ and u, ω are odd integers.*

(i) *If $s \geq a + 3$, then for every $k \in \mathbb{Z}_2$,*

$$v_2(N - qk^2) = a, \quad 2^{-a}(N - qk^2) \equiv r \pmod{8}.$$

In particular, every nonnegative integral residual $N - qk^2$ remains in $\mathcal{E}_{\varepsilon, r}$.

(ii) *If $s \leq a + 2$ and $N \geq 2^{\max(4, a)} q$, then there is an integer $k \geq 1$ such that*

$$0 \leq N - qk^2 \quad \text{and} \quad N - qk^2 \notin \mathcal{E}_{\varepsilon, r}.$$

Proof. For (i), qk^2 is divisible by 2^{a+3} , so subtraction preserves the valuation and the odd part modulo 8.

For (ii), put $e = s - a$. When $-2 \leq e \leq 2$, the following choices suffice. The final column identifies the condition defining $\mathcal{E}_{\varepsilon, r}$ that fails.

e	k	$N - qk^2$	change in the residual
2	1	$2^a(u - 4\omega)$	$u_2 \equiv r - 4 \pmod{8}$
1	1	$2^a(u - 2\omega)$	$u_2 \equiv r - 2\omega \not\equiv r \pmod{8}$
0	2	$2^a(u - 4\omega)$	$u_2 \equiv r - 4 \pmod{8}$
-1	1	$2^{a-1}(2u - \omega)$	$v_2 = a - 1 \not\equiv \varepsilon \pmod{2}$
-2	4	$2^a(u - 4\omega)$	$u_2 \equiv r - 4 \pmod{8}$.

Every parenthesized factor is odd, so the displayed valuations and odd parts are exact. In the second row, $2\omega \equiv 2$ or $6 \pmod{8}$.

For $e \leq -3$, write $j = a - s \geq 3$. If j is odd, take $k = 1$:

$$N - q = 2^s(2^j u - \omega), \quad v_2(N - q) = s \not\equiv a \pmod{2}.$$

If j is even, compare $k = 1$ and $k_0 = 2^{(j-2)/2}$:

k	$v_2(N - qk^2)$	$u_2(N - qk^2) \pmod{8}$
1	s	$-\omega$
k_0	$a - 2$	$4 - \omega$.

The valuations have the same parity as a , but the two odd parts differ by 4, so at least one is not r modulo 8.

All choices satisfy $k^2 \leq \max(16, 2^a)$, since $s \geq 0$. The assumed size bound therefore makes the chosen residual nonnegative. The odd factors displayed above are nonzero, so it is in fact positive. This completes the proof. $\square$

The subtraction lemma represents a target once it is large enough and $v_2(q) \leq v_2(N) + 2$. Three tasks remain: establish the size bound, treat F_{22} directly for g , and classify the children for which the valuation condition can fail. The last task will require one additional coset of the orthogonal sublattice.

Lemma 5.2 (2-adic size bound). *For a 2-adic parent T with Fibonacci truant τ_T , put*

$$q_T^{\text{bd}} = \tau_T(\det T)^2.$$

Every Fibonacci exception $N > \tau_T$ of T satisfies

$$N \geq 2^{\max(4, v_2(N))} q_T^{\text{bd}},$$

except when $(T, N) = (g, F_{22})$.

Proof. First consider the family $4^j(8b+7)$, whose truant is F_{10} . The exceptions of valuation 0 have $n \equiv 10 \pmod{12}$, by Lemma 2.3. The next two are therefore F_{22} and F_{34} . For the three parents in this family, the bounds q_T^{bd} are 55, 880, and 4455, respectively, and

$$F_{22} > 16 \cdot 880, \quad F_{34} > 16 \cdot 4455.$$

This proves the assertion for exceptions of valuation 0, apart from (g, F_{22}) .

If an exception in this family has positive valuation, the valuation is even and at least 4: Lemma 2.2 excludes valuation 2. The first possible indices are 12 and 36, and F_{12} is not an exception. By (1), $2^{v_2(F_n)} \leq 4n$, while

$$\frac{F_{36}}{4 \cdot 36} > 4455.$$

The monotonicity of F_n/n now gives the bound for all these exceptions.

For the other two families, exceptions must have odd valuation. Valuation 1 occurs only at $n \equiv 3 \pmod{6}$, where the recurrence modulo 16 gives $F_n \equiv 2 \pmod{16}$ and hence $u_2(F_n) \equiv 1 \pmod{8}$. Such terms belong to neither family. Beyond the initial window in Lemma 2.4, the remaining indices through 90 with odd valuation at least 3 are

n	54	66	72	78	90
$v_2(F_n)$	3	3	5	3	3
$u_2(F_n) \pmod{8}$	1	3	3	5	7

Together with that initial window, this shows that the next exception after F_{42} in $4^j(16b+14)$ is F_{90} , and the next after F_{30} in $4^j(16b+10)$ is F_{78} . At these respective indices, each parent in the family satisfies both

$$F_n > 16q_T^{\text{bd}}, \quad \frac{F_n}{4n} > q_T^{\text{bd}}.$$

These inequalities persist by Lemma 2.1; together with (1), they imply the required bound. $\square$

To identify the children for which the valuation condition can fail, put

$$a_T = \min\{v_2(N) : N \in \mathcal{F} \cap E(T)\}.$$

Then

$$\begin{aligned} a_T &= 0 && \text{for } T = \langle 1, 1, 1 \rangle, \langle 1, 2, 2 \rangle, g, \\ a_T &= 3 && \text{for } T = \langle 1, 1, 2 \rangle, \langle 1, 2, 3 \rangle, \langle 1, 2, 4 \rangle. \end{aligned}$$

For the first line the truant 55 has valuation 0. For the second, the preceding proof excludes valuation 1, while the truants F_{30} and F_{42} both have valuation 3. In particular, $v_2(\tau_T) = a_T$ for every parent in this branch.

Call a child *high-valuation* if $v_2(q) \geq a_T + 3$. For every other child, $v_2(q) \leq v_2(N) + 2$ holds for every Fibonacci exception N of its parent. We use the explicit inequality for individual targets; a high-valuation child may satisfy it when $v_2(N)$ is large.

Combining the size and valuation bounds gives the following representation criterion.

Proposition 5.3 (Escape for a single Fibonacci target). *Let M be a child of a 2-adic parent T , and let $N > \tau_T$ be a Fibonacci exception of T . If*

$$v_2(q) \leq v_2(N) + 2, \quad N \geq 2^{\max(4, v_2(N))} q,$$

then M represents N . Consequently, every child that is not high-valuation represents all its parent's Fibonacci exceptions, with the possible exception of F_{22} when $T = g$.

Proof. Part (ii) of Lemma 5.1 gives a nonnegative residual outside the parent's positive exceptional family. Regularity and Lemma 4.1 represent N . For the consequence, use $q \leq q_T^{\text{bd}}$ and Lemma 5.2; the truant itself is represented by the definition of a child. $\square$

To handle the remaining target for g and the high-valuation candidates, we first record a local obstruction for odd d .

Lemma 5.4 (The obstruction for odd d). *If an extension candidate of a 2-adic parent has odd d and $v_2(q) \geq v_2(\tau_T) + 3$, it does not represent τ_T and is not a child. In particular, no candidate with $c = 0$ can be a high-valuation child.*

Proof. For odd d , we have $M_2 \cong T_2 \perp \langle q \rangle$. Part (i) of Lemma 5.1 says that $\tau_T - qk^2$ stays in the parent's locally obstructed square class for every $k \in \mathbb{Z}_2$. Thus M_2 cannot represent τ_T . When $c = 0$ we have $d = 1$, and $v_2(\tau_T) = a_T$. $\square$

The size estimate leaves one value to handle directly, for every child of g .

Lemma 5.5 (The remaining g target). *Every child of g represents $F_{22} = 17711$.*

Proof. The last standard basis vector generates $\mathbb{Z}^3/A\mathbb{Z}^3 \cong \mathbb{Z}/9\mathbb{Z}$, so take $c = (0, 0, c_3)$ with $0 \leq c_3 \leq 8$. Here $d \mid 9$ and $\delta \leq 55$. If $d \leq 3$, then $q \leq 495$. If $d = 9$, then $3 \nmid c_3$ and

$$q = 81m - 18c_3^2, \quad \delta = m - 2c_3^2/9.$$

Thus $9 \mid q$. The fractional part of $2c_3^2/9$ is at least $2/9$, so the integer $m \leq 55 + 2c_3^2/9$ gives $\delta \leq 55 - 2/9$ and $q \leq 4437$.

Since d is odd, Lemma 5.4 gives $v_2(q) \leq 2$.

Put $N = 17711 \equiv 7 \pmod{8}$. If q is even, its valuation is 1 or 2, and $N - q > 0$ is odd with residue different from 7 modulo 8. If q is odd and $q \leq N/4$, then $N - 4q > 0$ has residue 3 modulo 8. Each residual is represented by g . The only remaining case is odd $q > N/4$. It requires $d = 9$, and the only odd multiple of 9 in $(17711/4, 4437]$ is 4437. Now

$$N - q = 17711 - 4437 = 13274$$

has valuation 1 and is represented by g . Apply Lemma 4.1 in each case. $\square$

It remains to classify the high-valuation children. The need to use more than the orthogonal sublattice is visible in the candidate $T = \langle 1, 2, 2 \rangle$, $c = (0, 1, 1)$, $m = 55$: here $q = 216$ and

$$F_{22} - 216k^2 \equiv 7 \pmod{8}$$

for every integer k . This candidate is a child, as the construction below will show, but the trivial-coset test cannot represent F_{22} .

For $T = \langle 1, 2, 4 \rangle$, the remaining candidates have even d and require a parity argument.

Lemma 5.6 (No high-valuation child of $\langle 1, 2, 4 \rangle$). *The parent $\langle 1, 2, 4 \rangle$ has no high-valuation child.*

Proof. Take $c_2 \in \{0, 1\}$ and $c_3 \in \{0, 1, 2, 3\}$. For odd c_3 , the formula $q = 16m - 8c_2^2 - 4c_3^2$ gives $v_2(q) = 2$. For even c_3 and $c_2 = 1$, one has $v_2(q) = 1$. After the case $c = 0$ is excluded by Lemma 5.4, the only possibility with $v_2(q) \geq 6$ is therefore $c = (0, 0, 2)$. In this case $q = 4(m - 1)$, so $m \equiv 1 \pmod{16}$, and

$$Q(x, y, z, w) = x^2 + 2y^2 + (2z + w)^2 + (m - 1)w^2.$$

The truant satisfies $F_{42} \equiv 8 \pmod{16}$. If w is odd, then $(2z + w)^2$ is 1 or 9 modulo 16 and $(m - 1)w^2 \equiv 0 \pmod{16}$, whereas $x^2 + 2y^2$ is never 7 or 15 modulo 16. Thus Q cannot represent F_{42} . If $w = 2w'$, write instead

$$Q = x^2 + 2y^2 + 4(z + w')^2 + 4(m - 1)w'^2.$$

The final term is divisible by 64. Since $v_2(F_{42}) = 3$, subtracting it preserves the valuation and the odd part modulo 8, so the residual is still locally obstructed for $\langle 1, 2, 4 \rangle$. Neither parity of w permits a representation, and the candidate is not a child. $\square$

Proposition 5.7 (High-valuation children). *The only 2-adic parent with high-valuation children is $\langle 1, 2, 2 \rangle$. These children are Fibonacci-universal and, in reduced form, are exactly the fourteen lattices with*

$$c = (0, 1, 1), \quad m \equiv 3 \pmod{4}, \quad 3 \leq m \leq 55.$$

Proof. We first exclude the other five parents. For $\langle 1, 1, 1 \rangle$ and g , the index d is always odd, so Lemma 5.4 applies. For $\langle 1, 1, 2 \rangle$, the only nonzero reduced cross-vector is $c = (0, 0, 1)$; it gives $q = 4m - 2$ and $v_2(q) = 1$. Candidates with $c = 0$ are excluded by the same lemma, and Lemma 5.6 handles $\langle 1, 2, 4 \rangle$.

For $\langle 1, 2, 3 \rangle$, take $c_2 \in \{0, 1\}$ and $c_3 \in \{0, 1, 2\}$. If $c_2 = 1$, then $v_2(q) = 1$. The remaining nonzero cross-vectors are $(0, 0, 1)$ and $(0, 0, 2)$, with $d = 3$ and respective values

$$q = 3(3m - 1), \quad q = 3(3m - 4).$$

Their odd index, and that of the candidates with $c = 0$, excludes every high-valuation child by Lemma 5.4.

Now let $T = \langle 1, 2, 2 \rangle$. A reduced cross-vector with exactly one of c_2, c_3 nonzero gives $q = 4m - 2$, of valuation 1; candidates with $c = 0$ are already excluded. Thus only $c = (0, 1, 1)$ remains, with

$$q = 4(m - 1), \quad \delta = m - 1.$$

The bounds $0 < \delta \leq 55$ give $2 \leq m \leq 56$, and $v_2(q) \geq 3$ is equivalent to m being odd.

If $m \equiv 1 \pmod{4}$, the candidate omits 55. For even $w = 2r$, completing squares in y, z gives

$$Q = x^2 + 2(y + r)^2 + 2(z + r)^2 + 4(m - 1)r^2,$$

which cannot be 7 (mod 8). For odd w , use $w^2 \equiv 1 \pmod{8}$ to obtain

$$Q \equiv x^2 + m + 2y(y + w) + 2z(z + w) \pmod{8}.$$

Each of the last two terms is 0 or 4 (mod 8); since $m \equiv 1$ or 5 (mod 8), the sum again cannot be 7 (mod 8).

For $m \equiv 3 \pmod{4}$, $3 \leq m \leq 55$, a single construction proves both that the candidate is a child and that it represents all odd Fibonacci exceptions. At $w = 1$ its value is

$$Q(x, y, z, 1) = x^2 + (y + z + 1)^2 + (y - z)^2 + m - 1.$$

If $N \geq 55$ and $N \equiv 7 \pmod{8}$, then $N - (m - 1) > 0$ is 1 or 5 $\pmod{8}$, so it is a sum of three squares. Since it is also 1 $\pmod{4}$, any such representation has one odd coordinate and two even coordinates. Place an even coordinate last and write

$$N - (m - 1) = X^2 + Y^2 + 4Z^2.$$

The integers X, Y have opposite parity. An integral representing vector is therefore

$$x = 2Z, \quad y = \frac{X + Y - 1}{2}, \quad z = \frac{X - Y - 1}{2}, \quad w = 1.$$

Taking $N = 55$ proves that all fourteen candidates are children. These vectors have $w = 1$, whereas the orthogonal sublattice has even w ; the extra coset supplies the missing representations.

For each such child, $m - 1 \equiv 2 \pmod{4}$, hence $v_2(q) = 3$. A positive-valuation Fibonacci exception has valuation at least 4. It therefore satisfies the valuation hypothesis of Proposition 5.3, and Lemma 5.2 gives the size hypothesis. These targets, together with the odd exceptions just represented, exhaust the parent's Fibonacci exceptions. Since the child contains the parent, it is Fibonacci-universal. $\square$

Proposition 5.8 (2-adic closure). *Every child of each of the six 2-adic parents is Fibonacci-universal.*

Proof. For a child that is not high-valuation, Proposition 5.3 and Lemma 5.5 represent every Fibonacci exception of its parent. The high-valuation children are covered by Proposition 5.7. All other Fibonacci values are already represented by the parent. $\square$

5.2 Universal polynomials for the odd-prime parents

The two remaining regular parents are governed by 3 and 5. When the new vector pairs nontrivially with the norm- p basis vector, a fixed nonzero coset supplies all sufficiently large integers. The following consequence of their representation criteria gives a common construction.

Lemma 5.9 (Universal coset polynomials). *Let (p, α) be $(3, 1)$ or $(5, 2)$, and let $a \in \mathbb{Z}$ with $p \nmid a$. Then*

$$P_{p,a}(x, y, z) = x^2 + \alpha y^2 + pz^2 + 2az$$

represents every nonnegative integer.

Proof. For $r \geq 0$, the integer $p^2r + pa^2$ has valuation 1 at p and square unit part. By Proposition 3.2, write

$$p^2r + pa^2 = X^2 + \alpha Y^2 + pZ^2.$$

In both cases, $X^2 + \alpha Y^2 \equiv 0 \pmod{p}$ forces $p \mid X, Y$. Put $X = px$ and $Y = py$ and divide by p :

$$pr + a^2 = px^2 + \alpha py^2 + Z^2.$$

Thus $Z \equiv \pm a \pmod{p}$. After changing its sign, write $Z = pz + a$. Substitution and division by p give $r = P_{p,a}(x, y, z)$. $\square$

5.3 The parent $\langle 1, 1, 3 \rangle$

The parent $\langle 1, 1, 3 \rangle$ omits a positive integer precisely when its 3-adic valuation is odd and its unit part is 2 (mod 3). The coset lemma handles children with a nonzero cross-term. For the orthogonal children, a small subtraction changes the valuation parity or the unit class. We first locate the small Fibonacci exceptions.

Lemma 5.10 (3-adic initial window). *Among the Fibonacci numbers F_n with $1 \leq n < 40$, the exceptions of $\langle 1, 1, 3 \rangle$ are exactly F_{16} and F_{20} .*

Proof. By Lemma 2.2, the indices below 40 with positive odd 3-adic valuation are those in the following table. Iterating the Fibonacci recurrence gives

n	4	8	16	20	28	32	36
$v_3(F_n)$	1	1	1	1	1	1	3
$u_3(F_n) \bmod 3$	1	1	2	2	1	1	1.

Only F_{16} and F_{20} have exceptional unit part. The entry at $n = 36$ uses the recurrence modulo 81, or equivalently exact integers. $\square$

Proposition 5.11 ($p = 3$ closure). *Every child of $\langle 1, 1, 3 \rangle$ is Fibonacci-universal.*

Proof. Write a reduced child as

$$Q = x^2 + y^2 + 3z^2 + 2c_3zw + mw^2, \quad c_3 \in \{0, 1, 2\}.$$

It represents its parent's truant $F_{16} = 987$ by definition. Every later Fibonacci exception is at least $F_{20} = 6765$. The child bound gives $m \leq 987 + c_3^2/3$, hence $m \leq 988$. If $c_3 \neq 0$, then

$$Q(x, y, z, 1) = m + P_{3, c_3}(x, y, z).$$

By Lemma 5.9, this represents every integer at least m , and therefore every remaining Fibonacci exception.

Suppose $c_3 = 0$. Then $q = m \leq 987$ and $v_3(m) \leq 1$ by Lemma 4.2. If $3 \nmid m$, subtract m from each remaining exception; the residual is positive and prime to 3, so the parent represents it.

It remains to consider $m = 3b$, where $3 \nmid b$ and $b \leq 329$. Write a remaining exception as $N = 3^a u$, with a odd and $u \equiv 2 \pmod{3}$. At $N = F_{20}$ we have $u = 2255$. Every other remaining exception has index $n \geq 40$ by Lemma 5.10, so (1) and Lemma 2.1 give

$$u \geq \frac{F_n}{3n} \geq \frac{F_{40}}{120} > 2255 > 4b.$$

Thus $u > 4b$ in every case. Choose $t \in \{1, 2\}$ and set

$$k = 3^{(a-1)/2}t, \quad R = N - mk^2 = 3^a(u - bt^2) > 0.$$

If $b \equiv 1 \pmod{3}$, take $t = 1$; then $u - b \equiv 1 \pmod{3}$. If $b \equiv 2 \pmod{3}$, both $u - b$ and $u - 4b$ are divisible by 3, but their difference $3b$ is not divisible by 9. Choose t so that $v_3(u - bt^2) = 1$; then $v_3(R) = a + 1$ is even. In either case the parent represents R , so the child represents N . Together with the values already represented by the parent, these are all Fibonacci targets. $\square$

5.4 The parent $\langle 1, 2, 5 \rangle$

For $\langle 1, 2, 5 \rangle$, the obstruction is an odd 5-adic valuation with nonsquare unit part. We use Lemma 5.9 when the new vector pairs nontrivially with the norm-5 basis vector. For the remaining children, we subtract an orthogonal square to change the valuation or unit part. Only the target F_{20} will require a further use of the child's representation of the truant. The following residue lemma will also be used at the prime 7 in §6.

Lemma 5.12 (Odd-prime residue escape). *Let $p \geq 5$ be prime and let $u_0, q_0 \in \mathbb{F}_p^\times$. There is an integer t , with $|t| \leq (p-1)/2$, such that $u_0 - q_0 t^2$ is a nonzero square modulo p . If u_0 is a nonsquare, then $t \neq 0$.*

Proof. Write χ for the quadratic character of $\mathbb{F}_p$, extended by $\chi(0) = 0$. For $A \neq 0$, the equation $y^2 = t^2 - A$ is equivalent to $(t-y)(t+y) = A$, which has $p-1$ solutions. Hence $\sum_t \chi(t^2 - A) = -1$. Taking $A = u_0/q_0$ shows that the conic $x^2 + q_0 t^2 = u_0$ has

$$p + \sum_{t \in \mathbb{F}_p} \chi(u_0 - q_0 t^2) = p - \chi(-q_0) \geq p - 1$$

points. At most two have $x = 0$. Since $p-1 \geq 4$, some point therefore has $x \neq 0$. Choose the least absolute representative of its t -coordinate. If u_0 is a nonsquare, the equation rules out $t = 0$. $\square$

Lemma 5.13 (5-adic initial window). *The exceptional Fibonacci values of $\langle 1, 2, 5 \rangle$ with index below 35 are exactly*

$$F_{15} = 610, \quad F_{20} = 6765, \quad F_{30} = 832040.$$

Proof. An exception must have index divisible by 5. The index 25 is ruled out because $v_5(F_{25}) = 2$ is even. The remaining candidates below 35 are 5, 10, 15, 20, and 30. At these indices the valuation is 1, and the unit parts are 1, 1, 2, 3, 3 modulo 5, respectively. Thus exactly F_{15} , F_{20} , and F_{30} are exceptional. $\square$

Proposition 5.14 ($p = 5$ closure). *Every child of $\langle 1, 2, 5 \rangle$ is Fibonacci-universal.*

Proof. Let M be a child, with reduced form

$$Q(x, y, z, w) = x^2 + 2y^2 + 5z^2 + 2c_2 yw + 2c_3 zw + mw^2,$$

where $c_2 \in \{0, 1\}$, $c_3 \in \{-2, -1, 0, 1, 2\}$, and $m \leq 611$. The child represents its truant $F_{15} = 610$. By Lemma 5.13, every other Fibonacci exception of its parent is at least $F_{20} = 6765$. Let N be such an exception.

If $c_3 \neq 0$, set $w = 2$. Completing the square in y gives

$$Q(x, y, z, 2) = P_{5, 2c_3}(x, y + c_2, z) + 4m - 2c_2^2.$$

Here $5 \nmid 2c_3$, so Lemma 5.9 represents $N - (4m - 2c_2^2) > 0$, since $4m - 2c_2^2 \leq 2444 < 6765$. Thus M represents N .

Suppose now that $c_3 = 0$. The bound $\delta \leq 610$ gives $m \leq 610$, and the orthogonal parameter is

$$q = \begin{cases} m, & c_2 = 0, \\ 4m - 2, & c_2 = 1. \end{cases}$$

In particular, $0 < q < 2440$, and Lemma 4.2 gives $v_5(q) \leq 1$. If $5 \nmid q$, the positive residual $N - q$ is prime to 5 and is represented by the parent. The trivial-coset test (Lemma 4.1) then represents N by M .

We may therefore assume $q = 5b$, where $5 \nmid b$ and $b < 488$. We first show that a unit-part bound handles every target except possibly F_{20} . Write $N = 5^a u$, where a is odd and $u \equiv 2$ or $3 \pmod{5}$. Whenever $u > 4b$, Lemma 5.12 supplies $t \in \{\pm 1, \pm 2\}$ such that $u - bt^2$ is a nonzero square modulo 5. With $k = 5^{(a-1)/2}|t|$, the residual

$$N - qk^2 = 5^a(u - bt^2) > 0$$

has square unit part and is represented by the parent. Again Lemma 4.1 represents N by M .

For $N > F_{20}$, this size condition always holds. By Lemma 5.13, either $N = F_{30}$, with

$$u = 166408 > 4 \cdot 488,$$

or $N = F_n$ with $n \geq 35$, when (1) and Lemma 2.1 give

$$u \geq \frac{F_n}{n} \geq \frac{F_{35}}{35} > 4 \cdot 488.$$

For $N = F_{20}$ we have $a = 1$ and $u = 1353$, so the same argument works whenever $b \leq 338$.

The only case still open is $N = F_{20} = 6765$ and $338 < b < 488$. Here the fact that M represents the truant imposes an additional restriction on b . In this range $c_2 = 1$, since $c_2 = 0$ would give $q = m \leq 610$. Thus $q = 4m - 2 = 5b$, and $b \equiv 2 \pmod{4}$. Choose a representation $Q(x, y, z, w) = 610$. The identity

$$4Q(x, y, z, w) = (2x)^2 + 2(2y + w)^2 + 5(2z)^2 + qw^2$$

forces $w = \pm 1$: the parent omits 610, so $w \neq 0$, while $q > 610$ rules out $|w| \geq 2$. Consequently the parent represents

$$2440 - q = 5(488 - b).$$

This excludes $b \equiv 1 \pmod{5}$, since the displayed value would then have valuation 1 and nonsquare unit part $2 \pmod{5}$. Together with $5 \nmid b$, this leaves $b \equiv 2, 3, 4 \pmod{5}$.

We claim that $k = 1$ now works. The residual is $R = 6765 - 5b = 5(1353 - b) > 0$. If $b \equiv 2$ or $4 \pmod{5}$, its unit part is a square. If $b \equiv 3 \pmod{5}$ but $b \not\equiv 3 \pmod{25}$, then $v_5(R) = 2$. Finally, if $b \equiv 3 \pmod{25}$, the condition $b \equiv 2 \pmod{4}$ gives $b \equiv 78 \pmod{100}$. The range $338 < b < 488$ leaves only $b = 378$ and $b = 478$, giving respectively

$$R = 4875 = 5^3 \cdot 39, \quad R = 4375 = 5^4 \cdot 7.$$

The first has square unit part $39 \equiv 4 \pmod{5}$, and the second has even valuation. Thus the parent represents R in every case, and Lemma 4.1 represents N by M .

We have represented every Fibonacci exception of the parent. Since the child also contains the parent, it is Fibonacci-universal. $\square$

6 Closing the irregular branch

For the regular parents, leaving the local obstruction was enough to represent a residual. For f , an eligible integer may instead be represented by the other genus class $h = \langle 1, 1, 7 \rangle$. We overcome this ambiguity by arranging a square factor: for certain primes p , an explicit integral linear map transfers every value of $p^2 h$ to a value of f .

The bounds in Lemmas 4.1 and 4.2 show that every child of f contains $f \perp \langle q \rangle$ for some

$$q \in \mathcal{Q}_7 := \{s, 7s : 1 \leq s \leq 147, 7 \nmid s\}. \quad (4)$$

We have $|\mathcal{Q}_7| = 252$ and $\max \mathcal{Q}_7 = 1022$. The bounds above allow a smaller set for actual children, but this simple uniform set is convenient. We prove a statement for every coefficient in it.

Proposition 6.1 (Uniform Fibonacci tail). *For every $q \in \mathcal{Q}_7$, the form $f \perp \langle q \rangle$ represents F_n for every $n \geq 48$.*

We first explain the square transfer and formulate the congruence principle. We then construct the two required residue classes for Fibonacci targets and prove positivity. The only finite residual checks left after the positivity estimate involve $48 \leq n \leq 72$.

6.1 A square factor forces representation

For the Fibonacci argument, eligibility refers to the genus $\{f, h\}$. By Proposition 3.3, every eligible positive integer is represented by at least one of these two forms.

Let $\mathcal{P}_7$ be the set of the thirty-one primes in Table 3. None is 7, and the largest is 503. For each $p \in \mathcal{P}_7$, the same table gives an integral matrix B_p satisfying

$$B_p^\top G_f B_p = p^2 G_h, \quad G_h = \text{diag}(1, 1, 7). \quad (5)$$

Thus $f(B_p v) = p^2 h(v)$ for every integral vector v . For example,

$$B_2 = \begin{pmatrix} 0 & -2 & 0 \\ 0 & 0 & 4 \\ -1 & 0 & -1 \end{pmatrix} \quad \text{satisfies} \quad f(B_2 v) = 4h(v).$$

Each identity is verified by multiplying the displayed integer matrices. The primes are auxiliary choices: we need square transfers for each of them and enough residue classes to obtain the congruence cover below. No minimality of the prime set is asserted. The companion library includes the search for the transfer matrices and their independent verification; see §A.

Kneser's neighbour construction gives a structural source of such maps; see [6]. If lattices L_f, L_h , isometric to f, h respectively, are p -neighbours in the same rational quadratic space, their intersection has index p in each. Hence $pL_h \subseteq L_f$, and writing this inclusion in integral bases gives a matrix satisfying (5). For the chosen primes, the displayed matrix identities provide the complete verification needed here.

Lemma 6.2 (Square transfer). *If $R > 0$ is eligible and $p^2 \mid R$ for some $p \in \mathcal{P}_7$, then f represents R .*

Proof. Since $p \neq 7$, division by p^2 preserves both the 7-adic valuation and the square class of the unit part. Hence R/p^2 is eligible and is represented by f or h . In the first case, $R/p^2 = f(v)$ gives $R = f(pv)$. In the second, $R/p^2 = h(v)$ gives $R = f(B_p v)$ by (5). $\square$

This is the role of the auxiliary square: it makes the choice of genus class irrelevant. The next step is to arrange that square while keeping the residual eligible and positive.

6.2 The transfer and congruence principle

The following principle combines the square factor with a separate condition for eligibility. It does not depend on a recurrence or on the density of the targets. We state it generally, then apply it to f and h .

Let L, L' represent the two classes in a genus of positive-definite integral lattices; throughout this principle, eligibility refers to this genus. Let $\mathcal{P}$ be a finite nonempty set of primes such that, for each $p \in \mathcal{P}$, there is an integral map B_p satisfying

$$Q_L(B_p v) = p^2 Q_{L'}(v).$$

Assume also that, for each $p \in \mathcal{P}$, division of an eligible positive integer by p^2 , when integral, preserves eligibility. Fix an integer $\mu \geq 1$ coprime to every prime in $\mathcal{P}$.

Proposition 6.3 (Square transfer with congruences). *Under these assumptions, let $U, q \in \mathbb{Z}_{>0}$. Suppose some $p \in \mathcal{P}$ admits classes $k_p \pmod{p^2}$ and $k_\mu \pmod{\mu}$ such that*

$$U \equiv qk_p^2 \pmod{p^2},$$

and every positive residual $U - qk^2$ with $k \equiv k_\mu \pmod{\mu}$ is eligible. If

$$U > q \left\lfloor \frac{\mu p^2}{2} \right\rfloor^2,$$

then $L \perp \langle q \rangle$ represents U .

Proof. The Chinese remainder theorem gives $k \equiv k_p \pmod{p^2}$ and $k \equiv k_\mu \pmod{\mu}$ with $|k| \leq \lfloor \mu p^2 / 2 \rfloor$. Thus $R = U - qk^2 > 0$ is eligible and divisible by p^2 . By hypothesis, R/p^2 is eligible, so it is represented by L or L' . In the first case, scaling a representing vector by p represents R by L ; in the second, applying B_p does so. Adding the coordinate k proves the assertion. $\square$

The hypothesis that eligibility survives division by p^2 holds, for example, if L_p represents every element of $\mathbb{Z}_p$ for each $p \in \mathcal{P}$. At the other primes, division by p^2 preserves local representation because p is a unit. In our application, $L = f$, $L' = h$, $\mu = 7$, and $\mathcal{P} = \mathcal{P}_7$; Proposition 3.3 supplies the eligibility hypothesis. We next construct the two residue classes for the Fibonacci targets.

6.3 Two congruence conditions for the residual

First remove an even power of 7 from the Fibonacci target. For $n \geq 48$, put

$$a = \left\lfloor \frac{v_7(F_n)}{2} \right\rfloor, \quad U = F_n / 49^a. \quad (6)$$

Then $v_7(U) \leq 1$. We remove only an even power of 7 so that a representation of U scales back by multiplying every coordinate by 7^a : the identity $U = f(v) + qk^2$ gives

$$F_n = f(7^a v) + q(7^a k)^2. \quad (7)$$

We will choose k so that the positive residual $R = U - qk^2$ satisfies

$$7 \nmid R \quad \text{or} \quad v_7(R) = 1 \quad \text{and} \quad R/7 \equiv 1, 2, 4 \pmod{7}. \quad (8)$$

By Proposition 3.3, either alternative makes R eligible. The bounds $v_7(U), v_7(q) \leq 1$ allow us to impose this condition by choosing k modulo 7. Independently, to make p^2 divide R , we seek a solution of $qk^2 \equiv U \pmod{p^2}$.

For an odd prime $p \nmid q$, a nonzero solution modulo p lifts by Hensel's lemma to a solution modulo p^2 . For $p = 2$, the required congruence is already modulo $p^2 = 4$: take k even when $U \equiv 0 \pmod{4}$, or odd when $U \equiv q \pmod{4}$. Thus no lifting is needed in this case. The following periodic check supplies this congruence for every index. We state it for F_n and then explain why it remains valid for U . Positivity will be imposed after the two congruences are combined.

Certificate 6.4 (Congruence cover). *For every $q \in \mathcal{Q}_7$ and every $n \geq 0$, at least one of the following conditions holds:*

1. $F_n \bmod 4 \in \{0, q \bmod 4\}$;
2. for some odd $p \in \mathcal{P}_7$, one has $p \nmid q$ and $F_n \bmod p \in q(\mathbb{F}_p^\times)^2$.

Exact verification. Iterate the Fibonacci recurrence from $(F_0, F_1) = (0, 1)$ modulo 4 and modulo each odd $p \in \mathcal{P}_7$. The periods, listed in Table 3, all divide $\Pi = 332640$. Thus it suffices to check the $252\Pi = 83825280$ pairs

$$(q, n) \in \mathcal{Q}_7 \times \{0, \dots, \Pi - 1\}.$$

For each odd p , compute the nonzero square residues by squaring $1, \dots, p-1$ and test the displayed alternatives. Every pair passes. The program `generate_tail_certificates.py` performs this check using periodic Boolean words; `verify_congruence_cover.cpp` independently tests every pair directly. $\square$

The same cover holds for the normalized target U . Indeed, 49^a is 1 modulo 4 and is an invertible square modulo every odd $p \in \mathcal{P}_7$. Dividing F_n by 49^a therefore preserves each of the two alternatives. For odd p , the root is nonzero and $p \nmid q$, so its derivative $2qk$ is nonzero modulo p ; this is exactly the hypothesis needed for the lift. We have consequently found a prime $p \in \mathcal{P}_7$ and a class $k_0 \pmod{p^2}$ such that

$$qk_0^2 \equiv U \pmod{p^2}.$$

It remains to make the residual eligible. Since $p \neq 7$, choosing $k \pmod{7}$ will not interfere with this square divisibility.

Lemma 6.5 (Combining the congruences). *Let $q \in \mathcal{Q}_7$ and $U > 0$ with $v_7(U) \leq 1$. If $qk_0^2 \equiv U \pmod{p^2}$ for some $p \in \mathcal{P}_7$ and*

$$U > q \left\lfloor \frac{7p^2}{2} \right\rfloor^2,$$

then $f \perp \langle q \rangle$ represents U .

Proof. We choose $k \pmod{7}$ so that a positive residual $R = U - qk^2$ satisfies (8). If $7 \nmid q$, avoid the at most two roots of $qk^2 \equiv U \pmod{7}$; this ensures $7 \nmid R$. If $7 \mid q$ but $7 \nmid U$, every choice already has this property.

The remaining case has $q = 7s$ and $U = 7u$, where $7 \nmid su$. By Lemma 5.12, choose $k \pmod{7}$ so that $u - sk^2$ is a nonzero square. Then $v_7(R) = 1$ and $R/7$ has square residue, as required.

The chosen class modulo 7 makes every positive residual eligible. Together with the given class $k_0 \pmod{p^2}$, it therefore satisfies the hypotheses of Proposition 6.3 with $L = f$, $L' = h$, and $\mu = 7$. That proposition represents U . $\square$

For a concrete residual, take $n = 48$, $q = 1022$, and $k = 6$. Here $U = F_{48}$, and

$$R = F_{48} - 1022 \cdot 6^2 = 4807490184.$$

We have $4 \mid R$, while $R/7 = 686784312 \equiv 4 \pmod{7}$. Thus R has valuation 1 at 7 with square unit part, and B_2 transfers a representation of $R/4$ from h to f if needed. This illustrates the two conditions imposed on the residual.

6.4 Positivity and the remaining indices

Proof of Proposition 6.1. Fix $q \in \mathcal{Q}_7$ and $n \geq 48$, and normalize F_n as in (6). The congruence cover and its lift give the first hypothesis of Lemma 6.5. Because $q \leq 1022$ and $p \leq 503$, its size hypothesis follows from $U > B$, where

$$K = \left\lfloor \frac{7 \cdot 503^2}{2} \right\rfloor = 885531, \quad B = 1022K^2 = 801416785304142. \quad (9)$$

Although B is large, Fibonacci growth reaches it quickly even after normalization. If $8 \mid n$, the bound in (1) gives

$$49^a \leq 7^{v_7(F_n)} \leq \frac{7n}{8}, \quad U \geq \frac{8F_n}{7n}.$$

If $8 \nmid n$, then $U = F_n$ and the same lower bound holds. Since F_n/n increases, the single exact inequality

$$8F_{82} - 7 \cdot 82B = 30433091008315220 > 0$$

proves $U > B$ for every $n \geq 82$. We can lower the threshold further: Lemma 2.2 gives $49 \mid F_n$ if and only if $56 \mid n$. No index from 73 through 81 is divisible by 56, so $U = F_n$ there, and

$$F_{73} = 806515533049393 > B.$$

Thus $U > B$ for every $n \geq 73$. The preceding lemma represents U , and (7) represents F_n .

The remaining indices are $48 \leq n \leq 72$, which we handle by certifying small choices of k directly. For each of the $25 \cdot 252 = 6300$ pairs (n, q) , the file `tail_certificates.json` supplies $k \geq 0$ and $p \in \mathcal{P}_7$ for which

$$R = U - qk^2 > 0, \quad p^2 \mid R, \quad R \text{ satisfies (8).}$$

Exact substitution verifies every record; the largest k is 993. The verifier also checks that every required pair occurs exactly once. The square-transfer lemma represents R , and scaling again represents F_n . §B describes the certificate data and their independent verification. $\square$

The threshold 73, the modulus Π , and the bound B are sufficient parameters of this proof. They play a different role from the sharp criterion endpoint 42, which is forced by a ternary truant.

Proposition 6.6 (Closure for f). *Every child of f is Fibonacci-universal.*

Proof. Every child contains one of the orthogonal extensions in Proposition 6.1, so it represents F_n for $n \geq 48$. By Proposition 3.5, the parent already represents $F_1, \dots, F_{47}$ except for $F_8 = 21$; the child represents this truant by definition. $\square$

7 The criterion and its minimality

We can now complete the escalation argument. The branch results prove sufficiency; a four-square construction then shows why every truant is indispensable.

Proof of Theorem 1.1. Sufficiency. §3 shows that C is exactly the set of truants occurring before rank 4. Suppose that a form Q represents every element of C . Starting with the zero lattice inside Q , adjoin a Q -vector representing the current truant. By Lemma 3.6, each such vector raises the rank. The first three steps therefore reach one of the nine ternary parents in Proposition 3.1. Adjoining its truant and reducing gives a quaternary child as defined in §4.

Every such child is Fibonacci-universal: Proposition 5.8 handles the six 2-adic parents, Propositions 5.11 and 5.14 handle the other two regular parents, and Proposition 6.6 handles f . Since the child is contained in Q , the form Q is Fibonacci-universal. The converse is immediate because $C \subseteq \mathcal{F}$.

Minimality. The argument follows the extension principle of Kala–Krásenský–Romeo [10, Proposition 3.1], using the four-square construction of Kim–Lee–Oh [14, Proposition 2.1]. We include the proof for completeness. Fix $t \in C$ and take the parent P_t with Fibonacci truant t listed in Table 1. We will enlarge P_t to represent every integer greater than t while still omitting t . Set

$$V_t = P_t \perp \langle t+1, t+1, t+1, t+1 \rangle \perp \bigoplus_{j=t+1}^{2t+1} \langle j \rangle,$$

with all sums orthogonal. Every nonzero vector in the added summands has norm greater than t , so V_t still omits t .

Now let $N > t$. Choose the unique $j \in \{t+1, \dots, 2t+1\}$ congruent to N modulo $t+1$. This is the least integer greater than t in its residue class, so $j \leq N$ and $N-j = (t+1)m$ with $m \geq 0$. Write $m = a_1^2 + \dots + a_4^2$ by Lagrange’s four-square theorem. The four equal summands then represent $(t+1)m$, and the unary summand $\langle j \rangle$ supplies j . Thus V_t represents N . Since P_t already represents every Fibonacci number below t , the lattice V_t represents every Fibonacci number except t .

Any Fibonacci criterion set omitting t would therefore be represented by V_t without forcing Fibonacci-universality. Hence every criterion set contains every $t \in C$. Together with sufficiency, this proves that C is the unique minimal Fibonacci criterion set. $\square$

Finally, representing $F_1, \dots, F_{42}$ suffices because this interval contains C . The parent $\langle 1, 1, 2 \rangle$ represents all preceding Fibonacci values and omits F_{42} , so the endpoint 42 is sharp.

8 Concluding remarks

The transfer and congruence principle in Proposition 6.3 also suggests applications to other target sequences. Retain its auxiliary prime set $\mathcal{P}$ and modulus μ . For targets $N_n = s_n^2 U_n$ with $s_n, U_n \in \mathbb{Z}_{>0}$, take a finite nonempty set $\mathcal{Q}$ of positive coefficients q and seek the two residue classes for each U_n and q . If s_n is coprime to the auxiliary primes, a solution of $N_n \equiv qk_p^2 \pmod{p^2}$ gives one for U_n by replacing k_p with $s_n^{-1}k_p$. Periodicity of the original targets can therefore reduce the auxiliary-prime cover to a finite calculation. Valuation information is used separately to find the eligibility class modulo μ .

Once both congruence conditions have been established, an explicit lower bound for U_n tending to infinity gives a computable index beyond which

$$U_n > (\max \mathcal{Q}) \left\lceil \frac{\mu(\max \mathcal{P})^2}{2} \right\rceil^2.$$

The proposition handles that tail, representations scale back to N_n , and only finitely many initial targets remain. A successful cover and the eligibility classes are substantive inputs, even when periodicity and growth are known.

Lucas recurrences provide a natural setting for these arithmetic inputs. For example, let $t_0 = 0$, $t_1 = 1$, and $t_{n+2} = at_{n+1} + bt_n$, with coprime integers a, b , and suppose the characteristic roots satisfy $\alpha > \max(1, |\beta|)$. For $n \geq 1$, the terms $t_n = (\alpha^n - \beta^n)/(\alpha - \beta)$ are positive and grow exponentially. Modulo an integer m coprime to b , the pair recurrence is invertible on a finite set,

so it is periodic. For each fixed prime p , the valuation formulas of Sanna [21, Theorem 1.5] give $v_p(t_n) \leq v_p(n) + C_p$ for a constant C_p depending on the recurrence; if $p \mid b$, all positive-index terms are p -units. Thus removing square factors supported on a fixed finite set of primes costs at most a polynomial factor, and exponential growth survives normalization. These facts supply periodicity, valuation control, and growth. An application still requires its own parent forms, transfer maps, and congruence cover; the proposition does not assert a criterion for every Lucas sequence.

In the meantime, the Fibonacci criterion gives one further instance in which 42 is the answer to everything.

“The Answer to the Great Question... Of Life, the Universe and Everything... Is... Forty-two”

—Douglas Adams, *The Hitchhiker’s Guide to the Galaxy*

A Computational companion

The accompanying *Fibonacci Criterion* library, release 0.1.0, contains the exact verification programs, certificate data, and optional discovery tools used in this paper. It is distributed as `fibonacci-criterion-0.1.0.zip`, with Python package name `fibonacci_criterion`. The release number is independent of the manuscript revision. This section records the computational proof inputs and explains how to reproduce their verification.

A.1 Proof inputs and their verification

Table 2 lists the finite inputs to the irregular branch. The regular branches are proved directly in §5; the package retains their earlier exhaustive searches only as optional corroboration.

Table 2: Computational inputs to the irregular branch.

Input	Scope	Verification
Initial representations	$9 \leq n \leq 47$	Substitute the thirty-nine triples into f (Certificate 3.4).
Square transfers	$p \in \mathcal{P}_7$	Check all thirty-one identities $B_p^\top G_f B_p = p^2 G_h$ (Equation (5)).
Congruence cover	$\mathcal{Q}_7 \times \{0, \dots, 332639\}$	Test all 83825280 pairs (Certificate 6.4).
Finite residuals	$\{48, \dots, 72\} \times \mathcal{Q}_7$	Check all 6300 records and exact coverage (Proposition 6.1).

The initial witnesses and transfer matrices are also printed in Sections A and B. Each residual certificate records n, q, k, p , the residual R , and the scale 7^a in $F_n = 49^a U$. The verifier checks maximal removal of powers of 49, the identity $R = U - qk^2$, positivity, divisibility by p^2 , eligibility of both R and R/p^2 , and the identity that scales the representation back to F_n . It rejects repeated JSON keys and repeated index-coefficient pairs, and checks that the latter form exactly the required Cartesian product. Thus record counts alone are not used as evidence of coverage.

The full congruence cover is checked by a separate C++ program that iterates the Fibonacci recurrence and tests the complete auxiliary prime set. It imports neither a Python generator nor the stored covering subsets. The required audit also verifies the small-index witnesses, the modulo-49 obstructions, the least periods, and the endpoint arithmetic used in the positivity argument. These finite checks leave the general genus, local representation, regularity, and valuation inputs to the proofs and references given in the manuscript.

A.2 Organization and use

The source distribution separates the following components.

- `fibonacci_criterion/` contains the verification library and command-line interface. The certificate files in its `data/` subdirectory are `initial_representations.json`, `square_transfers.json`, and `tail_certificates.json`. The independent C++ verifier, `verify_congruence_cover.cpp`, is in `native/`.
- `tools/` contains `generate_square_transfers.py` and `generate_tail_certificates.py`. Both require an explicit output destination.

- **optional/** contains the earlier regular-branch enumerations and their witness data. These programs are not run by the required audit.
- **docs/** describes the verification protocol, certificate schemas, and Python interface; **paper/** contains this paper and its sources. **provenance/** preserves the certificate migration record and historical logs, while **validation/** records the checks for the present software release.

From the extracted distribution’s top-level directory, run

```
python3 -m fibonacci_criterion verify -output audit-output
```

This requires Python 3.9 or later and a C++17 compiler, with `g++` used by default. The option `-cxx` selects another compatible compiler. No installation is needed for this command; the distribution also includes an installable wheel and standard Python packaging metadata.

The audit writes logs and a machine-readable report to the selected output directory. The report records commands, exit codes, timings, source and input hashes, and confirmation that the supplied certificates were unchanged. Failed checks return a nonzero exit status. Python verification checks remain active under `-O` and `-OO`; the C++ verifier rejects builds that disable its assertions.

Certificate discovery and proof verification are separate operations. The required audit reads the supplied data without invoking a discovery program or replacing any certificate. Reconstructed certificate files can be submitted to the independent verifier through its Python interface. All proof checks use exact integer or rational arithmetic; timings are recorded only as performance information.

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Use of Generative AI

I used LLMs to assist with coding, computations, analysis, and revision, including GPT-6 Astra, GPT-5.6 Pro, and Claude Fable 5 (all accessed in part via Poe with the support of Quora, where I am an advisor). The problem, methods, and final written form are my own, and of course any errors remain my responsibility.

A Initial Fibonacci witnesses for f

The following triples verify Certificate 3.4 by substitution into $f(x, y, z) = x^2 + 2y^2 + 2yz + 4z^2$. The same data are stored in `initial_representations.json`. Their keys are precisely the indices $9, \dots, 47$.

n	(x, y, z)	n	(x, y, z)	n	(x, y, z)
9	(4, 3, 0)	22	(5, 93, 2)	35	(11, 2046, 178)
10	(3, 3, 2)	23	(3, 117, 5)	36	(4, 2630, 183)
11	(1, 6, 1)	24	(0, 132, 30)	37	(5, 3082, 602)
12	(0, 6, 3)	25	(3, 158, 49)	38	(5, 4074, 566)
13	(1, 6, 5)	26	(3, 130, 119)	39	(12, 5289, 568)
14	(3, 13, 1)	27	(2, 249, 86)	40	(7, 4265, 3132)
15	(12, 7, 8)	28	(7, 121, 240)	41	(33, 764, 6223)
16	(7, 19, 4)	29	(15, 491, 29)	42	(18, 11272, 557)
17	(7, 12, 15)	30	(2, 80, 433)	43	(13, 3348, 9335)
18	(26, 24, 9)	31	(15, 685, 191)	44	(11, 12992, 6827)
19	(3, 44, 3)	32	(49, 1007, 65)	45	(14, 22701, 1958)
20	(11, 56, 3)	33	(12, 1051, 368)	46	(3, 29831, 894)
21	(2, 57, 22)	34	(9, 1473, 322)	47	(9, 37714, 1550)

B Certificates for the irregular branch

Table 3 gives the matrices in (5) by their rows. The period is modulo 4 when $p = 2$, and modulo p otherwise. All periods divide 332640. The companion library verifies these identities and periods, and checks the full congruence cover independently; §A describes the programs and their roles.

The optional construction program `generate_tail_certificates.py` handles $48 \leq n \leq 72$ and $q \in \mathcal{Q}_7$. It removes the powers of 49 from F_n , obtaining U , then tests nonnegative k in increasing order and records the first positive residual $R = U - qk^2$ satisfying (8) and divisible by p^2 for some $p \in \mathcal{P}_7$. Every pair succeeds with $k \leq 993$. The file `tail_certificates.json` stores the resulting data, with one residual certificate for each pair in $\{48, \dots, 72\} \times \mathcal{Q}_7$. Its independent verification is part of the audit described in §A.

Table 3: Integral square transfers and Fibonacci periods.

p	Period	Row 1	Row 2	Row 3
2	6	$(0, -2, 0)$	$(0, 0, 4)$	$(-1, 0, -1)$
3	8	$(-1, -1, 7)$	$(-1, 2, 1)$	$(-1, -1, -2)$
5	20	$(-3, -3, 7)$	$(-2, 3, 3)$	$(-1, -1, -6)$
11	10	$(-3, -7, -21)$	$(-1, -6, 15)$	$(-5, 3, -2)$
13	28	$(-5, -9, -21)$	$(-6, -3, 19)$	$(-3, 5, -10)$
17	36	$(-1, -15, -21)$	$(-3, 6, -29)$	$(9, -1, 2)$
19	18	$(-3, -3, 49)$	$(-6, 13, 3)$	$(-7, -7, -6)$
23	48	$(-5, -21, 21)$	$(-6, 7, 39)$	$(-9, -1, -22)$
29	14	$(-7, -15, 63)$	$(-6, -17, -33)$	$(15, -1, 10)$
31	30	$(-15, -13, 63)$	$(-2, 21, 27)$	$(-13, -3, -26)$
41	40	$(-5, -9, 105)$	$(-3, -30, -19)$	$(21, 5, 10)$
43	88	$(-7, -15, -105)$	$(-15, -26, 33)$	$(-15, 17, -10)$
47	32	$(-11, -45, -21)$	$(-18, -1, 81)$	$(-15, 7, -50)$
53	108	$(-21, -39, 77)$	$(-14, 27, 69)$	$(-19, -5, -54)$
61	60	$(-3, -3, 161)$	$(-19, 42, 3)$	$(-23, -23, -6)$
71	70	$(-15, -63, -77)$	$(-27, -14, 117)$	$(-23, 17, -66)$
83	168	$(-9, -9, 217)$	$(-26, 57, 9)$	$(-31, -31, -18)$
89	44	$(-3, -87, 49)$	$(-21, 14, 165)$	$(-37, -5, -78)$
107	72	$(-35, -51, -231)$	$(-30, -59, 123)$	$(-39, 41, -22)$
109	108	$(-3, -105, -77)$	$(-41, -18, 183)$	$(-37, 13, -114)$
167	336	$(-33, -135, 245)$	$(-45, 74, 243)$	$(-65, -23, -150)$
181	90	$(-37, -177, 21)$	$(-39, 14, 345)$	$(-75, 13, -158)$
197	396	$(-35, -99, 441)$	$(-45, 126, 173)$	$(-81, -49, -122)$
199	22	$(-55, -183, 147)$	$(-75, 58, 309)$	$(-63, -7, -226)$
211	42	$(-15, -203, -147)$	$(-70, -33, 369)$	$(-77, 27, -206)$
241	240	$(-27, -237, -91)$	$(-66, -17, 447)$	$(-95, 23, -222)$
263	176	$(-87, -105, -595)$	$(-75, -154, 267)$	$(-95, 103, -30)$
281	56	$(-27, -133, -651)$	$(-102, -159, 257)$	$(-97, 105, -122)$
307	88	$(-89, -9, 777)$	$(-102, 207, -65)$	$(-105, -121, -94)$
379	378	$(-127, -105, -903)$	$(-111, -238, 303)$	$(-135, 151, 10)$
503	1008	$(-159, -471, -203)$	$(-158, -3, 915)$	$(-175, 89, -486)$

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