

FINITE-FAMILY AVOIDANCE AND ISOLATION GAPS FOR QUADRATIC FORMS

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ABSTRACT. An *isolation* of a positive definite integral lattice f represents every proper full-rank sublattice of f but not f itself. We prove that every isolation of f has rank at least $\text{rk } f + 2$.

In fact, every finite family of rank- $(\text{rk } f + 1)$ lattices that miss f must miss a common proper full-rank sublattice; this *finite-family avoidance* property passes to orthogonal sums. A spinor-genus condition gives the corresponding result for rank- $(\text{rk } f + 2)$ lattices when $\text{rk } f \geq 3$, and hence an isolation gap of at least 3 for every lattice with such a summand.

1. INTRODUCTION

An *isolation* of a positive definite integral lattice f is a lattice that represents every proper full-rank sublattice of f but does not represent f itself. Here representation means isometric embedding, and integrality means that the bilinear form takes integer values. A lattice *misses* another if it does not represent it. A *subform* of f is a lattice represented by f ; it is proper if it is not isometric to f . We write $\text{Iso}(f)$ for the least isolation rank when one exists, and call $\text{Iso}(f) - \text{rk } f$ the *isolation gap*. Using full-rank sublattices is equivalent to using all proper subforms (Remark 2.1).

We write $h \cong L$ for isometry and $h \perp L$ for orthogonal sum; $\langle a_1, \dots, a_n \rangle$ denotes the lattice with diagonal Gram matrix $\text{diag}(a_1, \dots, a_n)$. A lattice is *indecomposable* if it is not an orthogonal sum of two nonzero lattices.

Isolations exist for every indecomposable lattice [CO] and every cubic lattice $I_n = \langle 1, \dots, 1 \rangle$ of rank n [Oh]. Known bounds already show that

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isolation gaps can be large [JKKKO, Oh]:

$$\begin{aligned} \text{Iso}(I_1) &= 3, & \text{Iso}(I_2) &= 5, & \text{Iso}(I_3) &= 6, \\ \text{Iso}(I_n) &\gg_\varepsilon n^{3/2-\varepsilon} \quad (\varepsilon > 0). \end{aligned}$$

Ju, Kim, Kim, Kim, and Oh [JKKKO] proved that no binary lattice isolates a unary lattice and no ternary lattice isolates a binary one, using inert-prime descent and Gauss composition, respectively. We prove the corresponding bound in every rank—and improve it under an arithmetic hypothesis on an orthogonal summand.

Main results. The *genus* $\text{gen } h$ consists of the isometry classes of lattices isometric to h over the real numbers and over the q -adic integers for every prime q . Spinor genera give a finer equivalence relation, recalled in Section 2.1; we write $\text{spn } h$ for the spinor genus containing h . In rank at least 3, the spinor genera in $\text{gen } h$ form a finite elementary abelian 2-group S_h , with identity $\text{spn } h$ and every element having square 1; its group law is described in Section 5.1.

Two lattices in the same rational quadratic space are *p-neighbors* if their intersection has index p in each. Chenevier’s [Ch] theorem gives a positive integer N_h and a homomorphism

$$\delta_h : (\mathbb{Z}/N_h\mathbb{Z})^\times \longrightarrow S_h$$

that records the change of spinor genus under p -neighbors for primes $p \nmid N_h$. Here $(\mathbb{Z}/N_h\mathbb{Z})^\times$ is the group of invertible residue classes. Our main condition is

$$(S) \quad \delta_h(-1) = 1.$$

Condition (S) ensures that p -neighbors preserve the spinor genus at primes $p \equiv -1 \pmod{N_h}$. Chenevier’s theorem then supplies a neighbor isometric to h for every sufficiently large such prime. The condition holds whenever $\text{gen } h$ is a single spinor genus. Section 5.1 states the precise theorem and proves independence of the choices.

Theorem A (Isolation gaps). *Let f be a positive definite integral lattice of positive rank.*

- (1) *Every isolation of f has rank at least $\text{rk } f + 2$.*
- (2) *If $f \cong h \perp \kappa$, where $\text{rk } h \geq 3$ and h satisfies (S), then every isolation of f has rank at least $\text{rk } f + 3$.*

We also give a finite congruence test sufficient for (S). Write B_h for the bilinear form of h , $Q_h(x) = B_h(x, x)$ for its norm, and $\det h$ for the determinant of a Gram matrix. Let

$$d_h = \gcd\{Q_h(x) : x \in h\}, \quad \mathfrak{q}_h = Q_h/d_h, \quad A_h = 2B_h/d_h.$$

Then $\mathfrak{q}_h$ is a *primitive* integral homogeneous quadratic polynomial, meaning that its coefficients have gcd 1. The form A_h is its integral polar form. With

$\mathbb{F}_\ell = \mathbb{Z}/\ell\mathbb{Z}$ for a prime ℓ , put

$$b_h = 8 \prod_{\substack{\ell \mid \det A_h, \ell \text{ odd} \\ \text{rk}_{\mathbb{F}_\ell}(A_h \bmod \ell)=1}} \ell;$$

the product is over odd primes ℓ . Our explicit sufficient condition is

(P) The congruence $\mathfrak{q}_h(w) \equiv -1 \pmod{b_h}$ has a solution $w \in h$.

For $\text{rk } h \geq 3$, we prove that (P) implies (S), so Theorem A(2) applies whenever h satisfies (P). The implication is strict: I_3 satisfies (S), whereas three squares cannot sum to 7 modulo 8, so it fails (P). We also give a direct proof of the conclusion of Theorem A(2) under (P) using represented primes and reflections.

Our proof of Theorem A strengthens the exclusion of an isolation in a given rank to a statement about finite families. For an integer $c \geq 1$, we say that a rank- r lattice h has *finite-family avoidance up to c* , written FF_c , if every finite family of rank- $(r + c)$ lattices missing h misses a common proper full-rank sublattice of h . The parameter c measures the allowed increase in rank. As Proposition 2.4 shows, this is equivalent to allowing family members of any ranks at most $r + c$, which explains the “up to c ” terminology. For a one-member family, the condition excludes an isolation of that rank.

Theorem B (Finite-family avoidance).

- (1) Every nonzero positive definite integral lattice has property FF_1 .
- (2) Every positive definite integral lattice of rank at least 3 satisfying (S) has property FF_2 ; as (P) implies (S), the conclusion holds under (P) as well.
- (3) If h has property FF_c for some $c \geq 1$, then so does $h \perp \kappa$ for every lattice κ , and h has property FF_d for $1 \leq d \leq c$.

Adding an orthogonal summand whose shortest nonzero vector has sufficiently large norm preserves isolation (Lemma 2.3), so Theorem B implies Theorem A. The same local construction also gives a separation theorem with no spinor hypothesis.

Theorem C (Same-rank separation). *Let h be a positive definite integral lattice of positive rank, and let p be an odd prime with $p \nmid \det h$. Then h has an index- p sublattice L missed by every positive definite integral lattice $K \not\cong h$ of the same rank and determinant as h . In particular, L is missed by every other class in $\text{gen } h$.*

Overview of the argument. We prove the finite-family statement, Theorem B, and then deduce the isolation gaps of Theorem A. The main construction produces a proper sublattice of h that every member of a prescribed finite family misses. Theorem C follows from a second case of the same construction.

For an odd prime $p \nmid \det h$, suppose that h has a p -neighbor $\mathcal{N} \cong h$, and put $L = h \cap \mathcal{N}$. The only integral lattices containing L with index p are h and $\mathcal{N}$. Thus an embedding of L in a lattice K missing h cannot extend to an index- p overlattice inside K ; its reduction $L/pL \hookrightarrow K/pK$ is consequently injective. If $p \nmid \det K$, the ambient space is nondegenerate, whereas L/pL has a two-dimensional radical, the subspace orthogonal to the whole space. This forces $\text{rk } K - \text{rk } h \geq 2$; at equality, it also forces $-\det h \det K$ to be a square modulo p (Lemma 3.2).

For Theorem B(1), the rank constraint alone suffices. In rank at least 2, Weber's theorem supplies infinitely many prime values of $\mathfrak{q}_h$, and reflection in a representing vector produces a p -neighbor isometric to h . Choose such a prime away from the determinants of all lattices in the family. Their rank difference is only 1, so all miss L . For unary h , a separate argument chooses one prime inert in the imaginary quadratic fields attached to the binary lattices; reduction modulo that prime forces a descent to a representation of h , again a contradiction.

For Theorem B(2), we also use the square condition. Under (S), Chenevier's theorem supplies a neighbor isometric to h at every sufficiently large prime $p \equiv -1 \pmod{N_h}$. A further congruence restriction makes each negative determinant product $-\det h \det K^{(i)}$ a nonsquare modulo the same prime. Thus every member of the family misses L . Under (P), represented prime values and reflections produce a neighbor satisfying the same determinant conditions.

Theorem B(3) passes avoidance to $h \perp \kappa$. For a finite family missing $h \perp \kappa$, all copies of κ in its members yield finitely many orthogonal complements missing h . A common missed sublattice $h' \subsetneq h$ then makes $h' \perp \kappa$ a common missed sublattice of $h \perp \kappa$. Taking a one-member family then excludes an isolation at the stated rank; adjoining unary summands of large norm rules out smaller isolation ranks as well. This proves Theorem A.

Finally, Theorem C uses a vector $\bar{v} \in h/ph$ of nonzero norm and the inverse image L of its orthogonal hyperplane. This time h is the unique integral index- p overlattice of L (Lemma 3.1). An embedding of L in a lattice K of the same rank and determinant as h would make K another such overlattice, so $K \cong h$.

Related work. The isolation question is closely related to the theory of finite universality criteria. Kim, Kim, and Oh [KKO] proved that every set $\mathcal{S}$ of positive definite integral forms of bounded rank admits a finite *criterion set*, i.e., a finite subset whose representation forces representation of all of $\mathcal{S}$. They asked whether all inclusion-minimal criterion sets for a given $\mathcal{S}$ have the same cardinality.

We identify isometric lattices when discussing criterion sets. For the set $\mathcal{S}_f$ of all subforms of a fixed form f , the singleton $\{f\}$ is one minimal criterion set. Any other minimal criterion set consists of proper subforms of

f , so its existence shows that those proper subforms already force representation of f . Elkies, Kane, and Kominers [EKK] answered the Kim, Kim, and Oh [KKO] cardinality question in the negative using this construction. For $f = \langle 1, 1, 2 \rangle$, they showed that $\{\langle 1, 1 \rangle, \langle 2, 2, 2 \rangle\}$ is a second minimal $\mathcal{S}_f$ -criterion set [EKK, Theorem 2.1]. Both members are proper subforms of f , so a lattice representing all its proper subforms must represent f itself—equivalently, $f = \langle 1, 1, 2 \rangle$ has no isolation. Their higher-rank examples use forced orthogonality between represented summands. For example, representing both E_8 and $\langle 1 \rangle$ forces $E_8 \perp \langle 1 \rangle$ [EKK, Proposition 3.1], where E_8 is the positive definite lattice of rank 8 with even norms and determinant 1.

Motivated in part by the [EKK] examples, Kim, Lee, and Oh [KLO] studied when representation of every proper subform of f forces representation of f . They called such a form *recoverable*; the finiteness theorem makes recoverability equivalent to the existence of a criterion set consisting of proper subforms. Oh [Oh] introduced the term *isolation* for a lattice witnessing failure of this implication, together with the minimum-rank invariant $\text{Iso}(f)$. Thus a form admits an isolation exactly when $\{f\}$ is the unique minimal $\mathcal{S}_f$ -criterion set.

Kim, Lee, and Oh [KLO] constructed isolations for indecomposable forms in at most four variables, and Chan and Oh [CO] extended existence to every indecomposable lattice. Oh [Oh] proved existence for all cubic lattices; meanwhile, rank-5 isolations of I_2 were studied by Ju, Kim, Kim, Kim, and Oh [JKKKO2]. Our finite-family result strengthens the exclusion of an isolation in a prescribed rank. The simultaneous formulation makes the general inheritance argument possible: The copies of an orthogonal summand produce finitely many complements, which must all miss one common sublattice.

Chan and Oh [CO, Proposition 3.1] also rule out isolations of the same rank by a related good-prime construction. Theorem C gives simultaneous separation at each odd prime not dividing the determinant, for lattices of equal rank and determinant.

The arithmetic inputs are Weber’s prime-representation theorem [We, Br] and Chenevier’s neighbor equidistribution theorem [Ch]. The higher-rank prime-representation theorem of Martin [Ma] and the stronger progression theorem of [Ko] provide related context.

Organization of the paper. Section 2 establishes our framework and introduces the formal definition of finite-family avoidance. Section 3 proves the prime-index lemma, Theorem C, and the obstruction to representing neighbor intersections. Sections 4 and 5 prove the avoidance bounds, with the spinor condition (S) followed by the congruence test (P). Section 6 gives examples, limitations, and open problems.

2. FINITE-FAMILY AVOIDANCE

2.1. Notation and conventions. Throughout, a *lattice* is a free $\mathbb{Z}$ -module of finite rank with a positive definite symmetric bilinear form B taking values in $\mathbb{Z}$. Its norm is $Q(x) = B(x, x)$, and its determinant is the determinant of a Gram matrix. We allow the zero lattice as an orthogonal summand, but consider isolation and finite-family avoidance only for positive-rank lattices. We write $\ell \hookrightarrow L$ for an isometric embedding, $\ell \cong L$ for isometry, and $L \perp \kappa$ for the orthogonal sum.

The unary lattice $\langle a \rangle$ has norm ax^2 . A lattice is *unimodular* if its determinant is 1, and *even* if all its norms are even; otherwise it is *odd*. For a positive integer a , let $L(a)$ denote L with bilinear form aB_L . Thus $I_n(a) = \langle a \rangle^{\perp n}$, while the sublattice $aL \subseteq L$ is isometric to $L(a^2)$. We write

$$\min L = \min\{Q(x) : 0 \neq x \in L\}.$$

Recall the primitive polynomial $\mathbf{q}_h = Q_h/d_h$ defined in the Introduction. The integer d_h is also the gcd of the diagonal entries and twice the off-diagonal entries of a Gram matrix. The integral polar form of $\mathbf{q}_h$ is

$$A_h(x, y) = \mathbf{q}_h(x + y) - \mathbf{q}_h(x) - \mathbf{q}_h(y) = 2B_h(x, y)/d_h.$$

The bilinear form B_h/d_h need not be integral; the reflection and prime-representation arguments use the polynomial $\mathbf{q}_h$ and its polar form A_h .

In our definitions of isolation and avoidance, the sublattices required to be represented or missed are proper and full-rank. Elsewhere sublattices may have any rank. The *saturation* of $M \subseteq K$ is $(M \otimes \mathbb{Q}) \cap K$; M is *primitive* if it equals its saturation, equivalently if K/M is torsion-free. An embedding is primitive if its image is primitive. A nonzero vector is primitive if its unary span is primitive, equivalently if it is not an integer multiple of another lattice vector by an integer of absolute value greater than 1. An *integral overlattice* of M is an integral lattice between M and $M \otimes \mathbb{Q}$ of finite index over M .

Remark 2.1 (Full-rank sublattices suffice). The definition of isolation we use is equivalent to Oh's definition [Oh, Definition 2.1], which requires representation of every proper sublattice, without restriction on rank; see also [JKKKO2, JKKKO, §1]. Indeed, if $M \subseteq f$ has rank less than $\text{rk } f$, then $M + pf$ is a proper full-rank sublattice for every prime p : Its image in f/pf is spanned by fewer than $\text{rk } f$ vectors. Hence representing every proper full-rank sublattice also forces representation of every lower-rank subform. Conversely, a proper full-rank sublattice M is not isometric to f , since

$$\det M = [f : M]^2 \det f > \det f.$$

The reduction to full-rank test sublattices also appears in [Oh, proof of Theorem 3.2] and [JKKKO2, §2], where isolation is verified by testing all prime-index sublattices.

For a prime q , let $\mathbb{Z}_q$ and $\mathbb{Q}_q$ denote the q -adic integers and numbers, and put $h_q = h \otimes \mathbb{Z}_q$. We write $[h]$ for the isometry class of h and $O(h)$ for its integral orthogonal group. To specify the refinement of the genus $\text{gen } h$ into spinor genera, take $[L] \in \text{gen } h$ and realize L and h in the same rational quadratic space. Their classes lie in the same *spinor genus* if there is a rational isometry σ such that $L_q = \sigma u_q(h_q)$ at every prime q , where each isometry u_q over $\mathbb{Q}_q$ has determinant 1 and trivial spinor norm. The spinor norm of a product of reflections is the product of the norms of their reflecting vectors, taken modulo squares in $\mathbb{Q}_q$; reflection in a vector v of nonzero norm sends x to $x - 2B(x, v)v/Q(v)$. We write $\text{spn } h$ for the set of classes in the spinor genus of h (see [OM, Ca, Ki, CS]). A *proper* isometry has determinant 1. Unless stated otherwise, our rational and integral isometries are ordinary isometries, with no orientation restriction.

The p -neighbor relation defined in the introduction is Kneser's [Kn]. At odd primes $p \nmid \det h$, a p -neighbor of h lies in $\text{gen } h$.

2.2. Avoidance and orthogonal sums.

Definition 2.2 (Finite-family avoidance up to c). For an integer $c \geq 1$, a lattice h of rank $r \geq 1$ has *finite-family avoidance up to c* , written FF_c , if every finite family $K^{(1)}, \dots, K^{(m)}$ of rank- $(r+c)$ lattices missing h misses a common proper full-rank sublattice $h' \subsetneq h$.

Note that we allow $m = 0$ in Definition 2.2: Any proper full-rank sublattice, such as $2h$, serves for the empty family. In the case of a one-member family $\{K\}$, finite-family avoidance says precisely that K does not isolate h .

Lemma 2.3 (Large orthogonal summands). *Let h have positive rank and basis $x_1, \dots, x_r$. If $E \neq 0$ and $\min E > \max_j Q_h(x_j)$, then*

$$h \hookrightarrow K \perp E \iff h \hookrightarrow K.$$

Consequently, if K isolates h , then so does $K \perp E$, and every larger isolation rank is possible.

Proof. The E -component of the image of each x_j must vanish, since a nonzero component would already have norm greater than $Q_h(x_j)$. Thus every embedding into $K \perp E$ takes values in K . The larger lattice still represents every proper sublattice represented by K and, by this equivalence, still misses h . Taking $E = \langle N \rangle^{\perp d}$ for sufficiently large N gives every larger isolation rank. $\square$

Proposition 2.4 (Inheritance and monotonicity). *If h has property FF_c , then $h \perp \kappa$ has property FF_c for every lattice κ , including the zero lattice. Moreover, the definition of FF_c is unchanged if each member of the family may have any rank at most $\text{rk } h + c$. In particular, h has FF_d for every $1 \leq d \leq c$.*

Proof. Let $F^{(1)}, \dots, F^{(m)}$ be lattices of rank $\text{rk } h + \text{rk } \kappa + c$ that all miss $h \perp \kappa$. There are only finitely many copies R of κ in these lattices: Each basis vector must map to a vector of a fixed norm, and positive definiteness makes each such set finite. The orthogonal complement $R^\perp$ inside the corresponding $F^{(i)}$ has rank $\text{rk } h + c$ and misses h , since a representation of h in $R^\perp$ would represent $h \perp \kappa$ in the corresponding lattice. Property FF_c supplies a proper full-rank $h' \subsetneq h$ missed by every complement. Each $F^{(i)}$ therefore misses $h' \perp \kappa$: An embedding would give a copy R of κ and a representation of h' in $R^\perp$.

For the rank assertion, let each $K^{(i)}$ have rank at most $\text{rk } h + c$ and miss h . Adjoin to it enough orthogonal unary summands of sufficiently large norm to bring its rank to $\text{rk } h + c$. Lemma 2.3 ensures that each enlarged lattice still misses h . A common sublattice missed by the enlarged family is also missed by the original family. This proves both the equivalent formulation and monotonicity. $\square$

Remark 2.5 (Integer-valued forms and scaling). An integer-valued quadratic form h may have a *half-integral* Gram matrix: Its diagonal entries are integral and its off-diagonal entries lie in $\frac{1}{2}\mathbb{Z}$. Doubling its norm gives an even lattice, denoted $h(2)$. Simultaneous scaling preserves embeddings and submodules.

In fact, h has FF_c among integer-valued forms if and only if $h(2)$ has FF_c among all integral lattices under our convention. Indeed, for any integral lattice K , the even sublattice

$$K^{\text{ev}} = \{x \in K : Q_K(x) \equiv 0 \pmod{2}\}$$

has index at most 2, since Q_K is additive modulo 2. Every embedding of a sublattice of $h(2)$ into K lands in K^{ev} . Replacing each representing lattice by its even sublattice and dividing all forms by 2 proves the equivalence. The same argument identifies the possible isolation ranks in the two conventions.

Scaling preserves $\mathfrak{q}_h$ and (P); for ranks r and $r + 2$, doubling multiplies the determinant product by the square 2^{2r+2} , preserving the quadratic fields used in Section 5. It also preserves (S): The spinor norm on determinant-1 isometries is unchanged, since an even number of reflection norms acquires a square factor. Spinor genera therefore identify under scaling, as do neighbor shifts at primes not dividing the scale. Our isolation-gap bounds thus apply to integer-valued forms as well.

3. PRIME-INDEX SUBLATTICES

Let p be an odd prime. For a bilinear space V over $\mathbb{F}_p$, the *radical* is

$$\text{rad}(V) = \{x \in V : B(x, V) = 0\};$$

the space V is *nondegenerate* if its radical is zero. A nonzero vector is *isotropic* if its norm is zero, and *anisotropic* otherwise; a space is anisotropic if it has no nonzero isotropic vector. A *hyperbolic plane* is a two-dimensional space with Gram matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ in a suitable basis.

The determinant of a nondegenerate space is understood modulo the subgroup $\mathbb{F}_p^{\times 2}$ of nonzero squares; the zero-dimensional space has determinant 1. We write $\left(\frac{a}{p}\right)$ for the Legendre symbol, equal to 0 if $p \mid a$, to 1 if a is a nonzero square modulo p , and to -1 otherwise.

Lemma 3.1 (Prime-index sublattices). *Let h have positive rank, let p be an odd prime with $p \nmid \det h$, and choose $0 \neq \bar{v} \in h/ph$. For any lift $v \in h$, put*

$$L = \{x \in h : B(x, v) \equiv 0 \pmod{p}\}, \quad R = \text{rad}(L/pL).$$

Then $[h : L] = p$, and the following assertions hold.

- (1) *If $Q(\bar{v}) \neq 0$, then $\dim R = 1$, and h is the unique integral index- p overlattice of L in $L \otimes \mathbb{Q}$.*
- (2) *If $Q(\bar{v}) = 0$, then $\dim R = 2$, and L has exactly two integral index- p overlattices, namely h and a p -neighbor $\mathcal{N}$ of h . Moreover,*

$$\det((L/pL)/R) \equiv -\det h \pmod{\mathbb{F}_p^{\times 2}}.$$

Proof. Put $V = h/ph$ and $W = \bar{v}^\perp$. Since V is nondegenerate, W has codimension 1, so $[h : L] = p$. Reduction gives an exact sequence

$$0 \longrightarrow ph/pL \longrightarrow L/pL \longrightarrow W \longrightarrow 0.$$

The kernel has dimension 1, and the form on L/pL is pulled back from W . As

$$\text{rad}(W) = W \cap \mathbb{F}_p \bar{v},$$

its dimension is 0 or 1 according as $Q(\bar{v})$ is nonzero or zero. This proves the assertions about R .

Let H be an integral index- p overlattice of L . For $x \in H$, we have $y = px \in L$ and $B(y, L) \subseteq p\mathbb{Z}$, whence

$$\bar{y} \in W \cap W^\perp;$$

if $Q(\bar{v}) \neq 0$, this intersection is zero. Thus $y \in ph$ and $x \in h$. It follows that $H \subseteq h$, and the indices give $H = h$.

Suppose now that $Q(\bar{v}) = 0$. We may choose a lift $w = v + pz \in h$ with $p^2 \mid Q(w)$: The required congruence

$$Q(v)/p + 2B(v, z) \equiv 0 \pmod{p}$$

is solvable because p is odd and $B(\bar{v}, \cdot)$ is nonzero. The lattice

$$\mathcal{N} = L + \mathbb{Z}(w/p)$$

is integral, since $B(w, L) \subseteq p\mathbb{Z}$ and $Q(w/p) \in \mathbb{Z}$. Also $w \in L$ and $w \notin ph$, so $[\mathcal{N} : L] = p$. As $w/p \notin h$, the prime-order group $\mathcal{N}/L$ meets h/L trivially, giving $\mathcal{N} \cap h = L$. Thus $\mathcal{N}$ is a p -neighbor of h .

If $H \neq h$, choose $x \in H \setminus h$; its class generates H/L , and $px \bmod ph$ is a nonzero multiple of $\bar{v}$. Multiplying x by an integer prime to p preserves this generating property and lets us arrange $px \equiv w \pmod{ph}$. Hence

$$x = w/p + z, \quad z \in h.$$

Integrality gives

$$Q(x) - Q(w/p) - Q(z) = \frac{2B(w, z)}{p} \in \mathbb{Z}.$$

Since p is odd and $\bar{w} = \bar{v}$, we obtain $z \in L$, and therefore $H = \mathcal{N}$.

Finally, choose $\bar{u} \in V$ with $B(\bar{v}, \bar{u}) = 1$ and replace $\bar{u}$ by $\bar{u} - \frac{1}{2}Q(\bar{u})\bar{v}$. The vectors $\bar{v}, \bar{u}$ then span a hyperbolic plane, whose Gram determinant is -1 . Its orthogonal complement identifies with $W/\mathbb{F}_p\bar{v}$, and hence with $(L/pL)/R$. This gives the asserted determinant square class. $\square$

Proof of Theorem C. The nondegenerate space h/ph contains a vector $\bar{v}$ of nonzero norm, as otherwise polarization would make its bilinear form zero. Let L be the corresponding index- p sublattice of Lemma 3.1. Suppose that $\varphi : L \hookrightarrow K$ embeds L in a lattice of the same rank and determinant as h , with image M . Then

$$[K : M]^2 \det K = \det L = p^2 \det h,$$

so $[K : M] = p$. Write $\varphi_{\mathbb{Q}}$ for the rational extension of φ . Then $H = \varphi_{\mathbb{Q}}^{-1}(K)$ is an integral index- p overlattice of L . The anisotropic case of Lemma 3.1 gives $H = h$, so $K \cong h$. $\square$

The isotropic case instead constrains representations in larger lattices.

Lemma 3.2 (Representations of neighbor intersections). *Let h have rank $r \geq 2$, let p be an odd prime with $p \nmid \det h$, and let $\mathcal{N}$ be a p -neighbor isometric to h . Put $M = h \cap \mathcal{N}$. Suppose that a lattice K of rank $r + c$, where $c \geq 0$, satisfies*

$$M \hookrightarrow K, \quad h \not\hookrightarrow K, \quad p \nmid \det K.$$

Then $c \geq 2$. If $c = 2$, then $-\det h \det K$ is a nonzero square modulo p .

Proof. The hyperplane $M/ph \subset h/ph$ equals $\bar{v}^{\perp}$ for some $\bar{v} \neq 0$. Since M has the two distinct integral index- p overlattices h and $\mathcal{N}$, Lemma 3.1 places it in the isotropic case: These are its only such overlattices, and both are isometric to h .

Identify M with its image in K , and let $S = (M \otimes \mathbb{Q}) \cap K$ be its saturation. If $p \mid [S : M]$, the finite group S/M has a subgroup of order p , giving an integral index- p overlattice of M inside K . Such an overlattice is isometric to h , a contradiction. Thus $p \nmid [S : M]$, so multiplication by p is bijective on S/M . It follows that inclusion induces an isometry

$$M/pM \cong S/pS.$$

Since S is primitive in K , reduction embeds S/pS in $V = K/pK$. Write W for its image and $R = \text{rad}(W)$. The isotropic case of Lemma 3.1 gives

$$\dim R = 2, \quad \det(W/R) \equiv -\det h \pmod{\mathbb{F}_p^{\times 2}}.$$

The space V is nondegenerate, so $\dim W^{\perp} = c$. As $R \subseteq W^{\perp}$, we have $c \geq 2$.

If $c = 2$, then $W^\perp = R$ and $W = R^\perp$. We may split off two hyperbolic planes whose isotropic lines span R . Indeed, for a basis e_1, e_2 of R , choose f_1 with $B(e_1, f_1) = 1$ and adjust it to have norm zero as in Lemma 3.1. Replacing e_2 by $e_2 - B(e_2, f_1)e_1$ puts it in the orthogonal complement of this first hyperbolic plane; repeating the construction there gives the second. The remaining orthogonal complement identifies with $R^\perp/R = W/R$. Each hyperbolic plane has determinant -1 , and therefore

$$(1) \quad \det K \equiv (-1)^2 \det(W/R) \equiv -\det h \pmod{\mathbb{F}_p^{\times 2}};$$

this is equivalent to the claimed square condition. $\square$

Write $\mathbb{H}$ for a hyperbolic plane over $\mathbb{F}_p$. In the codimension-2 case, non-degenerate quadratic spaces over $\mathbb{F}_p$ are classified by their dimension and determinant square class, so (1) is equivalent to

$$K/pK \cong (h/ph) \perp \mathbb{H}.$$

The lemma therefore gives a geometric obstruction to representing a neighbor intersection: in codimension 2, the ambient space modulo p must be a hyperbolic stabilization of h/ph .

More precisely, let h have rank $r \geq 2$, and let $K^{(1)}, \dots, K^{(m)}$ have rank $r + 2$ and miss h . Call p a *common good odd prime* if

$$p \nmid 2 \det h \prod_{i=1}^m \det K^{(i)}.$$

Suppose that, at such a prime, h has a p -neighbor $\mathcal{N} \cong h$. If none of the ambient spaces modulo p is a hyperbolic stabilization of h/ph , that is, if

$$K^{(i)}/pK^{(i)} \not\cong (h/ph) \perp \mathbb{H} \quad (1 \leq i \leq m),$$

then Lemma 3.2 implies that the proper full-rank sublattice $h \cap \mathcal{N}$ is missed by every $K^{(i)}$.

In arithmetic terms, these hypotheses are

$$(2) \quad \begin{aligned} & p \nmid 2 \det h \prod_{i=1}^m \det K^{(i)}, \\ & \mathcal{N} \cong h, \\ & \left(\frac{-\det h \det K^{(i)}}{p} \right) = -1 \quad (1 \leq i \leq m). \end{aligned}$$

Thus a self-neighbor of h at a common good odd prime produces a common missed sublattice whenever all the ambient spaces modulo p have the wrong isometry type.

4. FF_1 , UNCONDITIONALLY

For a quadratic field of fundamental discriminant D , let χ_D denote its primitive quadratic Dirichlet character, of conductor $|D|$. For an odd prime $p \nmid D$, we have $\chi_D(p) = \left(\frac{D}{p}\right)$; when $D < 0$, we have $\chi_D(-1) = -1$. The prime p is *inert* precisely when $\chi_D(p) = -1$.

4.1. Unary lattices. Whereas in rank at least 2 we may construct p -neighbors isometric to h at infinitely many primes and apply Lemma 3.2, the unary case requires a separate argument because its only rational isometries are ± 1 .

Proposition 4.1 (Unary avoidance). *Every unary lattice has property FF_1 . More precisely, if binary lattices $K^{(1)}, \dots, K^{(m)}$ all miss $\langle u \rangle$, then there are infinitely many primes q such that all of them miss $\langle q^{2a}u \rangle$ for every $a \geq 1$.*

Proof. The case of the empty family is immediate. Otherwise, let $D_i < 0$ be the fundamental discriminant of $\mathbb{Q}(\sqrt{-\det K^{(i)}})$ and put $\mathfrak{m} = \text{lcm}_i |D_i|$. Dirichlet's theorem gives infinitely many primes $q \equiv -1 \pmod{\mathfrak{m}}$ with $q \nmid 2 \prod_i \det K^{(i)}$. For every such prime,

$$\chi_{D_i}(q) = \chi_{D_i}(-1) = -1 \quad \text{for every } i.$$

Thus q is inert in all these fields. Each $K^{(i)}/qK^{(i)}$ is an anisotropic binary space: A nondegenerate binary form over $\mathbb{F}_q$ is isotropic exactly when the negative of its determinant is a square. If $Q(v) = q^{2a}u$ in $K^{(i)}$, anisotropy gives $v \in qK^{(i)}$. Dividing by q and repeating a times gives a vector of norm u , a contradiction. The sublattice $\langle q^{2a}u \rangle \subset \langle u \rangle$ has index q^a . $\square$

The proof of Proposition 4.1 is essentially the inert descent used for a single lattice in [JKKKO, Proposition 2.3(1)], but applied at one common prime for the whole family.

4.2. Prime values and reflections. We use Weber's theorem that every positive definite binary polynomial $AX^2 + BXY + CY^2$ with integral coefficients and $\gcd(A, B, C) = 1$ represents infinitely many primes [We]; see also [Br, §1]. A binary restriction gives both the higher-rank consequence and the control of square classes needed below.

Lemma 4.2 (Prime values in unit square classes). *Let $\mathfrak{q}$ be a primitive positive definite integral homogeneous quadratic polynomial on a free $\mathbb{Z}$ -module H of rank at least 2. Suppose that $\mathfrak{q}$ represents a modulo an integer $N \geq 1$, where $\gcd(a, N) = 1$. Then $\mathfrak{q}$ represents infinitely many primes p satisfying*

$$p \equiv at^2 \pmod{N} \quad \text{for some } t \in \mathbb{Z} \text{ with } \gcd(t, N) = 1.$$

In particular, $\mathfrak{q}$ represents infinitely many primes.

Proof. Choose $u \neq 0$ with $\alpha = \mathfrak{q}(u) \equiv a \pmod{N}$. For each prime $\ell \mid \alpha$, primitivity gives $w_\ell \in H/\ell H$ with $\mathfrak{q}(w_\ell) \neq 0$: Evaluations on coordinate

vectors and their pairwise sums detect all coefficients, including when $\ell = 2$. The Chinese Remainder Theorem gives w_0 with these residues. Adding a suitable multiple of $\prod_{\ell|\alpha} \ell$ times a vector independent of u , we obtain w independent of u with $\gcd(\alpha, \mathfrak{q}(w)) = 1$.

The positive definite binary polynomial

$$F(X, Y) = \mathfrak{q}(Xu + NYw)$$

is primitive, since its diagonal coefficients α and $N^2\mathfrak{q}(w)$ are relatively prime. Weber's theorem supplies infinitely many prime values of F . For those not dividing N , reduction gives $p \equiv aX^2 \pmod{N}$ with $\gcd(X, N) = 1$. Taking $N = 1$ gives the final assertion. $\square$

Lemma 4.3 (Neighbors from reflections). *Let h have rank at least 2, let p be odd with $p \nmid \det h$, and suppose that $\mathfrak{q}_h(v) = p$ for some $v \in h$. Put*

$$L_{p,v} = \{x \in h : A_h(x, v) \equiv 0 \pmod{p}\}.$$

The rational reflection

$$s_v(x) = x - \frac{2B_h(x, v)}{Q_h(v)}v = x - \frac{A_h(x, v)}{p}v$$

satisfies $h \cap s_v(h) = L_{p,v}$ and $[h : L_{p,v}] = p$. Thus $s_v(h)$ is a p -neighbor isometric to h .

Proof. The equality $\mathfrak{q}_h(v) = p$ makes v primitive. Also $p \nmid d_h$: Otherwise every entry of a Gram matrix would be divisible by p , since p is odd. Thus $A_h \bmod p$ is a nonzero scalar multiple of the nondegenerate form $B_h \bmod p$. As $v \notin ph$, the functional $A_h(-, v)$ is nonzero modulo p , giving $[h : L_{p,v}] = p$.

For $x \in h$, primitivity of v gives $s_v(x) \in h$ if and only if $p \mid A_h(x, v)$. Since s_v is an involutive isometry, this is precisely the condition that $x \in h \cap s_v(h)$. Applying s_v to the intersection gives $[s_v(h) : L_{p,v}] = p$ as well. $\square$

Proof of Theorem B(1). Proposition 4.1 proves the assertion when $\text{rk } h = 1$. For $\text{rk } h \geq 2$, let $K^{(1)}, \dots, K^{(m)}$ be rank- $(\text{rk } h + 1)$ lattices missing h . By Lemma 4.2, we may choose an odd prime

$$p \nmid \det h \prod_i \det K^{(i)}$$

represented by $\mathfrak{q}_h$. Lemma 4.3 gives a p -neighbor $\mathcal{N} \cong h$, and then Lemma 3.2 shows that every $K^{(i)}$ misses the proper index- p sublattice $h \cap \mathcal{N}$. $\square$

5. FF_2 UNDER (S) & (P)

We now prove FF_2 under (S), then give the sufficient test (P) and a direct reflection proof under that test. By Lemma 3.2, it suffices to find a p -neighbor $\mathcal{N} \cong h$ satisfying the quadratic-character conditions (2).

5.1. The spinor criterion. Let h have rank at least 3. Recall that S_h is the finite elementary abelian 2-group of spinor genera in $\text{gen } h$, with identity $\text{spn } h$ and multiplication induced by composition of local isometries. For representatives L and M in the common rational space $h \otimes \mathbb{Q}$, choose g_q, k_q with $g_q(h_q) = L_q$ and $k_q(h_q) = M_q$. The product of the spinor genera of L and M is the spinor genus of the lattice with completions $g_q k_q(h_q)$. These local isometries may be chosen to be the identity at all but finitely many primes. The product is independent of the representatives and choices; this is the group of [Ch, (5.5)], formed using the groups $O(h_q)$ of all local isometries preserving h_q .

We use the following consequence of Chenevier's neighbor equidistribution theorem.

Theorem 5.1 (Chenevier, specialized). *There exists an integer $N_h \geq 2$, divisible by $2 \det h$, along with a homomorphism*

$$\delta_h : (\mathbb{Z}/N_h \mathbb{Z})^\times \longrightarrow S_h$$

such that:

- (1) *For every prime $p \nmid N_h$, every p -neighbor changes the spinor-genus label by $\delta_h(p)$.*
- (2) *For every sufficiently large prime $p \nmid N_h$ with $\delta_h(p) = 1$, the lattice h has a p -neighbor isometric to h .*

Proof. The spinor shift and its dependence on a residue class are [Ch, §5.7, (R2), and Remark 5.11]. Take N_h to be a multiple of Chenevier's modulus and of $2 \det h$, also divisible by the finitely many primes excluded in his construction, and pull back the shift map to this modulus.

Remark 5.10 of [Ch] makes Theorem 5.9 there applicable to the full local orthogonal stabilizers and ordinary isometry classes; we apply it with both class variables equal to $[h]$. As p tends to infinity through primes with trivial shift, the proportion of neighbors isometric to h tends to

$$\frac{|\mathcal{O}(h)|^{-1}}{\sum_{[L] \in \text{spn } h} |\mathcal{O}(L)|^{-1}} > 0;$$

hence, the neighbor count is positive for every sufficiently large such prime. $\square$

The spinor hypothesis we use is

$$(S) \quad \delta_h(-1) = 1.$$

Condition (S) is independent of the chosen modulus and shift map. Dirichlet's theorem gives a prime $p \equiv -1$ modulo the least common multiple of any two choices. Since h/ph is nondegenerate of rank at least 3, it has a nonzero isotropic vector and hence a p -neighbor by Lemma 3.1. Both maps record the spinor shift of this same neighbor, so their values at -1 agree. In particular, (S) holds whenever $\text{gen } h$ comprises a single spinor genus.

Remark 5.2 (Inheritance of (S)). Condition (S) also passes to orthogonal sums. Indeed, if h satisfies (S), put $f = h \perp \kappa$ and choose a sufficiently large prime

$$p \equiv -1 \pmod{\text{lcm}(N_h, N_f)}.$$

Theorem 5.1 supplies a p -neighbor $\mathcal{N} \cong h$, so $\mathcal{N} \perp \kappa$ is a p -neighbor isometric to f . Thus $\delta_f(-1) = \delta_f(p) = 1$.

The summand hypothesis in Theorem A(2) is therefore equivalent to requiring $\text{rk } f \geq 3$ and (S) for f itself. The gap bound can also be deduced by applying Theorem B(2) directly to f , together with Lemma 2.3. Proposition 2.4 gives inheritance of FF_c for every $c \geq 1$, without an arithmetic hypothesis.

Proof of Theorem B(2) under (S). For the empty family, the sublattice $2h \subsetneq h$ suffices. Let $K^{(1)}, \dots, K^{(m)}$ be a nonempty finite family of rank- $(\text{rk } h + 2)$ lattices missing h . For each i , let D_i be the fundamental discriminant of the imaginary quadratic field $\mathbb{Q}(\sqrt{-\det h \det K^{(i)}})$. Put

$$\mathfrak{m} = \text{lcm}(N_h, |D_1|, \dots, |D_m|).$$

Choose a sufficiently large prime $p \equiv -1 \pmod{\mathfrak{m}}$ avoiding all the determinants. Then $\delta_h(p) = \delta_h(-1) = 1$, so Theorem 5.1 gives a p -neighbor isometric to h . Moreover,

$$\left(\frac{-\det h \det K^{(i)}}{p} \right) = \chi_{D_i}(p) = \chi_{D_i}(-1) = -1 \quad (1 \leq i \leq m),$$

since each D_i is negative. Lemma 3.2 now gives a proper full-rank sublattice missed by every member of the family. $\square$

5.2. An explicit sufficient congruence test. We first identify when $\mathfrak{q}_h$ represents -1 modulo every positive integer. Lemma 4.2 then supplies represented primes in the corresponding unit square class, from which we deduce (S).

Put

$$b_h = 8 \prod_{\substack{\ell | \det A_h, \ell \text{ odd} \\ \text{rk}_{\mathbb{F}_\ell}(A_h \bmod \ell) = 1}} \ell;$$

again the product is over odd primes. Our explicit sufficient condition for (S) is

$$(P) \quad \mathfrak{q}_h(w) \equiv -1 \pmod{b_h} \quad \text{for some } w \in h.$$

Proposition 5.3 (The congruence test). *For a lattice h of rank at least 2, condition (P) holds if and only if $\mathfrak{q}_h$ represents -1 modulo every integer $N \geq 2$. In rank at least 3, these conditions imply (S).*

Proof. One direction takes $N = b_h$. Conversely, suppose (P) holds. At an odd prime ℓ , the reduction of $\mathfrak{q}_h$ has the same rank as $A_h \bmod \ell$, which is positive by primitivity. If its rank is 1, then $\ell \mid b_h$, and (P) gives a

representation of -1 modulo ℓ . Otherwise, diagonalization gives a binary restriction $cx^2 + dy^2$ with $c, d \neq 0$. The sets

$$\{cx^2 : x \in \mathbb{F}_\ell\}, \quad \{-1 - dy^2 : y \in \mathbb{F}_\ell\}$$

each have $(\ell+1)/2$ elements, so they intersect and again give a vector w with $\mathfrak{q}_h(w) \equiv -1 \pmod{\ell}$. In either case, apply Hensel's lemma to $\mathfrak{q}_h(tw) + 1$ at $t = 1$: Its derivative $2\mathfrak{q}_h(w)$ is a unit modulo ℓ , so scalar multiples of w represent -1 modulo every ℓ^e .

At 2, condition (P) gives w with $\mathfrak{q}_h(w) \equiv -1 \pmod{8}$. The 2-adic unit $-\mathfrak{q}_h(w)^{-1}$ is 1 modulo 8, hence a square modulo every power of 2. Scaling w gives a representation modulo every 2^e . The Chinese Remainder Theorem, applied to coordinates in a basis of h , combines these representations modulo any N .

Now suppose $\text{rk } h \geq 3$. The equivalence just proved and Lemma 4.2 give a represented prime $p \equiv -t^2 \pmod{N_h}$ with $\gcd(t, N_h) = 1$. Since N_h is divisible by $2 \det h$, Lemma 4.3 gives a p -neighbor isometric to h . Theorem 5.1(1) and the fact that every element of S_h has square 1 give

$$1 = \delta_h(p) = \delta_h(-1)\delta_h(t)^2 = \delta_h(-1). \quad \square$$

5.3. A direct proof from (P). It is also possible to give a direct proof of FF_2 under (P) and determine exactly when the represented-prime reflection construction works for every finite family.

Let $\text{rk } h \geq 2$. For a fixed nonempty family of rank- $(\text{rk } h + 2)$ lattices $K^{(i)}$ missing h , let D_i be the fundamental discriminant of the imaginary quadratic field $\mathbb{Q}(\sqrt{-\det h \det K^{(i)}})$, and put $\mathfrak{m} = \text{lcm}_i |D_i|$. There are a prime p represented by $\mathfrak{q}_h$ and a p -neighbor $\mathcal{N}$ satisfying (2) if and only if there is a unit class $a \pmod{\mathfrak{m}}$ represented by $\mathfrak{q}_h$ with $\chi_{D_i}(a) = -1$ for every i .

Necessity follows by reduction modulo $\mathfrak{m}$. Conversely, Lemma 4.2 gives infinitely many represented primes in the unit square class of a modulo $\mathfrak{m}$. Quadratic characters are trivial on squares, so $\chi_{D_i}(p) = \chi_{D_i}(a) = -1$ for every i . We may therefore avoid the divisors of $2 \det h \prod_i \det K^{(i)}$ and apply Lemma 4.3.

Theorem 5.4 (The scope of the reflection argument). *Let h have rank $r \geq 2$. Condition (P) holds if and only if every nonempty finite family of rank- $(r + 2)$ lattices missing h admits a prime p represented by $\mathfrak{q}_h$ and a p -neighbor $\mathcal{N}$ satisfying (2). Under these conditions, h has property FF_2 .*

Proof. Suppose (P) holds. For any such family, Proposition 5.3 shows that $\mathfrak{q}_h$ represents -1 modulo $\mathfrak{m}$. Since $\chi_{D_i}(-1) = -1$ for every i , the preceding equivalence supplies the required prime. Equation (2) then gives FF_2 .

Conversely, suppose every such family admits the required prime, and write $b = b_h$. Consider the negative fundamental discriminants -4 , -8 , and

$$D_\ell = \begin{cases} -\ell & \ell \equiv 3 \pmod{4}, \\ -4\ell & \ell \equiv 1 \pmod{4}, \end{cases} \quad (\ell \mid b, \ell \text{ an odd prime}).$$

The corresponding characters have conductors whose least common multiple is b . Choose an integer t with $t^2 > \min h$. For each of these discriminants, the rank- $(r+2)$ lattice

$$K_D = \langle t^2 \rangle^{\perp(r+1)} \perp \langle t^2 | D | \det h \rangle$$

misses h because its minimum is t^2 , and

$$-\det h \det K_D = D(t^{r+2} \det h)^2.$$

The assumed prime for this family therefore gives a represented unit class $a \pmod{b}$ with $\chi_D(a) = -1$ for every chosen D .

The conditions $\chi_{-4}(a) = \chi_{-8}(a) = -1$ force $a \equiv 7 \pmod{8}$. For each odd $\ell \mid b$, quadratic reciprocity gives

$$\chi_{D_\ell}(a) = \begin{cases} \left(\frac{a}{\ell}\right) & \ell \equiv 3 \pmod{4}, \\ \chi_{-4}(a) \left(\frac{a}{\ell}\right) & \ell \equiv 1 \pmod{4}; \end{cases}$$

since these values are all -1 , we have

$$\left(\frac{a}{\ell}\right) = \left(\frac{-1}{\ell}\right).$$

Thus $-a$ is a square modulo 8 and every odd prime dividing b . The Chinese Remainder Theorem gives $a = -u^2 \pmod{b}$ for a unit u . Scaling a vector representing a by an integer inverse of u modulo b gives a representation of $-1 \pmod{b}$, proving (P). $\square$

This equivalence concerns reflections in vectors v with $\mathbf{q}_h(v) = p$, used with the determinant obstruction of Lemma 3.2; it does not characterize FF_2 . The example I_3 distinguishes the two, and Section 6.4 shows that no binary lattice satisfies (P).

Completion of the proofs of Theorems B and A. The arguments in Sections 4 and 5, together with Proposition 5.3, prove Theorem B(1)–(2). Theorem B(3) is Proposition 2.4.

Applied to one-member families, Theorem B excludes isolations of rank $\text{rk } f + 1$ for every f , and of rank $\text{rk } f + 2$ when $f \cong h \perp \kappa$ with h satisfying the hypotheses of part (2). In each case, an isolation of smaller rank can be enlarged to the excluded rank by Lemma 2.3. This proves Theorem A. $\square$

6. EXAMPLES, LIMITATIONS, AND QUESTIONS

In this section, we present examples that illustrate both the reach of our avoidance criteria and the limitations of the prime-index method. We first apply the congruence test to a ternary lattice whose isolation rank is at least 6. We then observed that the lattice I_3 separates (S) from (P), showing that an obstruction to the reflection construction need not obstruct finite-family avoidance. We also discuss the local obstruction and the difficulties in rank 2, and conclude with several open questions.

We use Hermite's bound

$$\min L \leq \gamma_n (\det L)^{1/n}$$

for a rank- n lattice L , where γ_n is the Hermite constant; in particular, $\gamma_2 = 2/\sqrt{3} < 2$ and $\gamma_3 = 2^{1/3} < 2$ [CS, Ch. 1].

6.1. A determinant-13 example. Consider the lattices with Gram matrices

$$f = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 5 \end{pmatrix}, \quad g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 13 \end{pmatrix}.$$

For either displayed Gram matrix, e_1, e_2, e_3 denote the corresponding basis vectors. We first apply the isolation bound to f , then compare its proper sublattices with the other class in its genus.

Proposition 6.1 (The isolation gap). *The lattice f is indecomposable and satisfies hypothesis (P). It therefore has property FF_2 , admits an isolation, and satisfies*

$$6 \leq \text{Iso}(f) < \infty.$$

Proof. A direct calculation gives $\det f = 13$. Its norm satisfies

$$Q(x, y, z) = (x + y + z)^2 + x^2 + y^2 + 4z^2.$$

Thus $\min f = 2$: For $z \neq 0$ the norm is at least 4, while for $z = 0$ the least positive value of $(x + y)^2 + x^2 + y^2$ is 2.

An orthogonal decomposition of f would have a unary summand $\langle d \rangle$ with $d \mid 13$, since determinants multiply under orthogonal sums. If $d = 1$, then f has a norm-1 vector. If $d = 13$, the complementary binary summand has determinant 1, so $\gamma_2 < 2$ forces its minimum to be 1. Both alternatives contradict the fact that $\min f = 2$, so f is indecomposable.

The diagonal norms 2 and 5 give $d_f = 1$, so $\mathfrak{q}_f = Q$ and

$$\mathfrak{q}_f(1, 1, 1) = 15 \equiv -1 \pmod{8}.$$

Moreover, $\det A_f = 2^3 \det f = 104$, and the upper-left 2×2 minor of A_f has determinant $12 \not\equiv 0 \pmod{13}$. Thus A_f has rank 2 modulo 13, so $b_f = 8$ and hypothesis (P) holds. Theorem B gives FF_2 , and Theorem A gives the lower bound $\text{Iso}(f) \geq 6$. Existence of an isolation follows from indecomposability and Chan and Oh's theorem [CO, Theorem 4.5]. Hence $6 \leq \text{Iso}(f) < \infty$. $\square$

The same lattice f also gives a refinement at $p = 2$ of Theorem C.

Proposition 6.2 (Separation at index 2). *The genus of f consists of the two classes $[f]$ and $[g]$. The lattice g represents none of the seven index-2 sublattices of f .*

Proof. In the Brandt–Intrau convention, a tuple (a, b, c, r, s, t) denotes

$$ax^2 + by^2 + cz^2 + ryz + szx + txy,$$

and its discriminant is -4 times the determinant of our Gram matrix. The Brandt–Intrau–Schiemann table of primitive odd ternary forms, at discriminant -52 , places precisely the two forms

$$(1, 1, 13, 0, 0, 0), \quad (2, 2, 5, 2, 2, 2)$$

in one genus [BI, BIS]; these are entries 34 and 36 of the electronic table and correspond to g and f . Proper and ordinary isometry classes coincide in odd rank, since $-I$ is an integral automorphism of determinant -1 . Hence $\text{gen } f = \{[f], [g]\}$, and their minima 2 and 1 distinguish the two classes.

For a lattice L , write

$$r_L(m) = \#\{x \in L : Q(x) = m\}.$$

The displayed expression for Q in the proof of Proposition 6.1 shows that $Q(x, y, z) = 2$ forces $z = 0$ and $x^2 + xy + y^2 = 1$. Thus the norm-2 vectors of f are exactly

$$\pm e_1, \quad \pm e_2, \quad \pm(e_1 - e_2).$$

Every index-2 sublattice $M \subset f$ is the inverse image of the kernel of a nonzero functional $\lambda : f/2f \rightarrow \mathbb{F}_2$; there are $2^3 - 1 = 7$ such functionals. The three opposite pairs reduce to the three nonzero vectors in $\langle e_1, e_2 \rangle_{\mathbb{F}_2}$. Therefore

$$r_M(2) \in \{2, 6\},$$

with value 6 when λ vanishes on this plane and 2 otherwise.

Suppose that M embeds in g , with image J . Since $\det M = 4 \det f = 52$ and $\det g = 13$, determinant comparison gives $[g : J] = 2$. Write J as the inverse image of $\ker \mu$ for a nonzero functional $\mu : g/2g \rightarrow \mathbb{F}_2$. As M has no norm-1 vector, neither does J , so the norm-1 vectors $\pm e_1, \pm e_2$ of g force

$$\mu(e_1) = \mu(e_2) = 1.$$

Consequently, all four norm-2 vectors $\pm e_1 \pm e_2$ of g lie in J . Thus $r_J(2) = 4$, contradicting $r_M(2) \in \{2, 6\}$. $\square$

The genus of f also comprises a single spinor genus, giving a direct verification of (S). Indeed, spinor genera in a genus have equal mass [Ch, Lemma 5.6], where the mass of a spinor genus is the sum of the class weights $|\mathcal{O}(L)|^{-1}$. If $[f]$ and $[g]$ were to lie in different spinor genera, they would therefore have equal automorphism-group orders. But $|\mathcal{O}(g)| = 16$: Its

norm-1 vectors force signed permutations of the first two coordinates, and the third coordinate can independently change sign. By contrast,

$$(x, y, z) \mapsto (y, -x - y - z, z)$$

is an automorphism of f of order 3, as the displayed expression for Q shows. Thus $3 \mid |\mathcal{O}(f)|$, a contradiction. The bound $\text{Iso}(f) \geq 6$ consequently also follows from Theorem A(2) under (S).¹

6.2. The two conditions on I_3 .

Proposition 6.3. *The lattice I_3 satisfies (S) but not (P). It has property FF_2 and fails FF_3 .*

Proof. Every integral unimodular ternary lattice L is isometric to I_3 : Since $\gamma_3 < 2$, it has a vector of norm 1, whose unary span splits off by integrality. The complement is binary and unimodular, so $\gamma_2 < 2$ gives another norm-1 vector, leaving I_1 .

Thus $\text{gen } I_3$ has one class, so (S) holds and Theorem B(2) gives FF_2 . Since $\text{Iso}(I_3) = 6$ [Oh], a rank-6 isolation gives a one-member family witnessing failure of FF_3 .

On the other hand, $b_{I_3} = 8$, and the odd values of $x^2 + y^2 + z^2$ modulo 8 are exactly 1, 3, 5. In particular, -1 is not represented modulo 8, so (P) fails. $\square$

For this lattice, the two congruence obstructions already occur in a family of two. The scaled cubic lattices

$$I_5(4) = \langle 4, 4, 4, 4, 4 \rangle, \quad I_5(2) = \langle 2, 2, 2, 2, 2 \rangle$$

miss I_3 because their minima exceed 1. Their negative determinant products with I_3 are -4^5 and -2^5 , giving the characters χ_{-4} and χ_{-8} , respectively. The simultaneous conditions

$$\chi_{-4}(a) = \chi_{-8}(a) = -1$$

force $a \equiv 7 \pmod{8}$, which three squares do not represent. Separately, the first condition holds at $a = 3$ and the second at $a = 5$, both represented by three squares.

6.3. Limits of the local obstruction.

Example 6.4 (Sharpness of the rank bound). In $h = I_2$, let $p = 5$ and let L_0 have basis $(1, 2), (-2, 1)$, with Gram matrix $\text{diag}(5, 5)$. This is $L_{5, (1, 2)} = h \cap s_{(1, 2)}(h)$ in Lemma 4.3. Its radical modulo 5 has dimension 2, and its only integral index-5 overlattices are isometric to I_2 , by Lemma 3.1.

¹The supplemental script `verify_lattices.py`, written with the assistance of Claude Opus 4.8 and revised with the assistance of GPT-6 Astra, uses exact arithmetic to check the displayed matrix, congruence, and norm-count calculations, including those in Proposition 6.3 and Example 6.4.

Consider the rank-4 lattice

$$K = \begin{pmatrix} 5 & 0 & 1 & 0 \\ 0 & 5 & 0 & 1 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{pmatrix}.$$

Its first two basis vectors span a primitive copy of L_0 . Permuting the basis expresses K as the orthogonal sum of two copies of $\begin{pmatrix} 5 & 1 \\ 1 & 2 \end{pmatrix}$. The identity

$$5a^2 + 2ab + 2b^2 = (a + b)^2 + (2a)^2 + b^2$$

gives minimum 2 for each block, so $\min K = 2$ and $\det K = 81$. Thus K misses I_2 but represents L_0 , even though $5 \nmid \det K$. This shows that the rank bound in Lemma 3.2 is sharp. Its square-class conclusion also holds: $-\det h \det K = -81 \equiv 4 \pmod{5}$.

Example 6.5 (Local congruence obstructions). For any $r \geq 2$, let

$$h = \langle 1 \rangle \perp \langle 3 \rangle^{\perp(r-1)}, \quad K = I_{r+2}(3).$$

All norms in K are divisible by 3, so K misses h . Here $-\det h \det K = -3^{2r+1}$, giving the character χ_{-3} . Every unit value of $\mathbf{q}_h = x_1^2 + 3x_2^2 + \cdots + 3x_r^2$ is 1 modulo 3 and hence has character +1, so no represented prime satisfies (2) for this family. Likewise, for $h = \langle 1 \rangle \perp \langle 4 \rangle^{\perp(r-1)}$ and $K = I_{r+2}(4)$, the character is χ_{-4} , while every odd value of $\mathbf{q}_h$ is 1 modulo 4.

These examples obstruct the represented-prime construction, not finite-family avoidance itself. Indeed, when $r \geq 3$, the displayed lattices h above have orthogonal summands $\langle 1, 3, 3 \rangle$ and $\langle 1, 4, 4 \rangle$, respectively. Each of these ternary lattices has a one-class genus [BIS, entries 20 and 49], so Theorem B gives FF_2 for both families in every rank at least 3.

6.4. Binary lattices.

Proposition 6.6. *No positive definite binary lattice satisfies (P).*

Proof. Suppose h has a p -neighbor $\mathcal{N} \cong h$ at an odd prime $p \nmid \det h$. As in the proof of Lemma 3.2, their intersection is the isotropic case of Lemma 3.1. In rank 2, its quotient by the radical is zero-dimensional and has determinant 1, so $-\det h$ is a square modulo p . Thus (2) requires every $\det K^{(i)}$ to be a nonsquare modulo p .

Choose an integer $t > \min h$ and let $K = I_4(t)$. It misses h and has square determinant t^4 , so no prime meets (2) for this family. The equivalence in Theorem 5.4 shows that h fails (P). $\square$

Thus extending our FF_2 approach to the binary case requires an obstruction that detects lattices of square determinant. A stronger prime-representation theorem alone would not suffice.

Remark 6.7 (Comparison with composition-law constraints). The single-isolation results of [JKKKO] are complementary. In their binary convention, let $\mathbf{n}f$ be the ideal generated by the values of f , let D_f be its classical

discriminant $(b^2 - 4ac$ for $f = ax^2 + bxy + cy^2)$, and let $f = kg$ denote scaling of the form. If a quaternary lattice F isolates a binary form f with $\mathfrak{n}f = s\mathbb{Z}$, then [JKKKO, Corollary 3.6] shows that the squarefree part of $\det F$ divides D_f/s^2 . If $f = kg$ with $\mathfrak{n}g = \mathbb{Z}$ and the Gauss class group of primitive positive definite binary forms of discriminant D_g , under proper equivalence, has no element of order 4, then [JKKKO, Corollary 3.7] makes $\det F$ a square. These conditions concern a single isolation, whereas Lemma 3.2 gives a condition at each prime usable across a finite family. The candidates in [JKKKO, Remark 3.8 and Table 1] are not proved to be isolations: The reported computation tests index- p sublattices only for $p \leq 149$.

6.5. Further questions. The finite-family formulation raises a question that is not settled by lower bounds for the rank of a single isolation.

Problem 6.8 (Finite families versus a single isolation). Can a positive-rank lattice h admitting an isolation fail FF_c for some

$$2 \leq c < \text{Iso}(h) - \text{rk } h?$$

In particular, does I_2 have property FF_2 ?

The equality $\text{Iso}(I_2) = 5$ excludes a single quaternary isolation, but does not settle avoidance for finite families of quaternary lattices. Our criteria do not decide this case: (P) fails in rank 2, the Chenevier input used here begins in rank 3, and inheritance from I_1 gives only FF_1 .

Problem 6.9 (Avoidance beyond the spinor condition). Does there exist a positive definite integral lattice of rank at least 3 that has property FF_2 but does not satisfy (S)?

Condition (S) supplies self-neighbors compatible with the simultaneous determinant obstructions. Failure of this condition need not preclude avoidance by other proper full-rank sublattices, so necessity for FF_2 would require an additional argument.

Problem 6.10 (The determinant-13 lattice). For the lattice f of Proposition 6.1, is $\text{Iso}(f) = 6$?

More generally, it would be useful to find explicit hypotheses guaranteeing FF_3 for lattices admitting isolations. Higher-index sublattices are a possible source of new obstructions, but a larger radical alone does not reproduce the saturation argument used here. Indeed, if $[h : L] = p^k$ with $k \geq 2$, then every index- p overlattice H of L has

$$\det H = p^{2k-2} \det h > \det h,$$

and therefore cannot represent h .

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