

# A HARMONIC LAW FOR RECIPROCAL ARITHMETIC FRACTALS

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ABSTRACT. Let  $M$  be a set of denominators in a field equipped with a multiplicative absolute value. Suppose that  $\#\{x \in M : |x| \leq T\} = T^{\alpha+o(1)}$ , where  $\alpha > 0$ , and that the ambient field has ball-covering dimension  $d \in (0, \infty)$ . We prove that the upper box dimension of  $M^{-1}$  is at most  $\alpha d/(\alpha + d)$ . Equality holds precisely when nearly  $T^\alpha$  denominators can be packed at separation  $T^{1-\alpha/d-o(1)}$  along arbitrarily large heights; the existence of box dimension at the same value is characterized by the corresponding all-scale condition. Arithmetic applications include distinct ideal norms, balanced algebraic integers, and definite quaternion orders. We also formulate the resulting finite-generation obstructions.

## 1. INTRODUCTION

Saito [Sai19] gave an elegant fractal proof of the infinitude of primes based on the classical box dimension identity

$$(1) \quad \dim_B\{1/n : n \geq 1\} = \frac{1}{2}.$$

For each prime  $p$ , the geometric ray  $\{p^{-k} : k \in \mathbb{Z}_{\geq 0}\}$  has box dimension 0, as does any finite setwise product of such rays. If only finitely many primes existed, then every reciprocal positive integer would belong to one such product, contradicting (1).

Only positivity of the dimension is needed for Saito's argument. But the exact value  $1/2$  in (1) suggests a quantitative question: *how do the growth and distribution of a given denominator set determine the box dimension of*

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*its reciprocals?* In this paper, we give a general upper bound and characterize equality by a pair of packing conditions.

**Covering numbers and box dimension.** For a subset  $E$  of a metric space and  $\varepsilon > 0$ , let  $N(E, \varepsilon)$  be the least number of sets of diameter at most  $\varepsilon$  needed to cover  $E$ . Set  $N(E, \varepsilon) = +\infty$  if no finite cover exists. For a nonempty totally bounded set, define the *upper and lower box dimension* as

$$\overline{\dim}_B E := \limsup_{\varepsilon \downarrow 0} \frac{\log N(E, \varepsilon)}{\log(1/\varepsilon)}, \quad \underline{\dim}_B E := \liminf_{\varepsilon \downarrow 0} \frac{\log N(E, \varepsilon)}{\log(1/\varepsilon)},$$

respectively; when these agree, their common value is the *box dimension*  $\dim_B E$ . We assign box dimension 0 to the empty set.<sup>1</sup>

**Denominator growth and ambient covering.** Let  $(K, |\cdot|)$  be a field equipped with a real-valued multiplicative absolute value satisfying the triangle inequality. We use the metric  $|x - y|$  and write

$$B(z, R) := \{x \in K : |x - z| \leq R\}.$$

Both Archimedean and non-Archimedean absolute values are allowed. For  $E \subset K^\times$ , put

$$E^{-1} := \{x^{-1} : x \in E\}.$$

When  $\dim_B(E^{-1})$  exists, we call it the *reciprocal dimension* of  $E$ .

Let  $M \subset K^\times$  satisfy  $|x| \geq 1$  for every  $x \in M$ . For  $T \geq 1$ , write

$$M(T) := \{x \in M : |x| \leq T\}, \quad A_M(T) := \#M(T).$$

We say that  $M$  has *counting exponent*  $\alpha > 0$  if every  $M(T)$  is finite and

$$(2) \quad A_M(T) = T^{\alpha+o(1)} \quad (T \rightarrow \infty);$$

equivalently, if  $\log A_M(T)/\log T \rightarrow \alpha$ .

We call  $s > 0$  an *admissible ball-covering exponent* for  $K$  if there is a constant  $C_s$  such that

$$(3) \quad N(B(z, R), r) \leq C_s \left(\frac{R}{r}\right)^s \quad (z \in K, 0 < r \leq R \leq 1);$$

let  $\mathcal{D}_K$  be the set of such exponents. The *ball-covering dimension* of  $K$  is then

$$d_K := \inf \mathcal{D}_K, \quad \inf \emptyset := +\infty.$$

Since  $\mathcal{D}_K$  is upward closed, every  $s > d_K$  is admissible whenever  $d_K < \infty$ ; the endpoint  $d_K$  need not be assumed admissible.

Throughout our main analysis, we assume that  $0 < d_K < \infty$  and write  $d := d_K$ . With the usual normalizations,  $d = 1$  for  $\mathbb{R}$ , for  $\mathbb{Q}_p$  with  $|p|_p = p^{-1}$ , and for  $\mathbb{F}_q((1/t))$  with  $|t| = q$ ; for  $\mathbb{C}$  with ordinary modulus,  $d = 2$ .

The ball-covering hypothesis implies that  $B(0, 1)$  is totally bounded. In fact, the reciprocal set  $M^{-1}$  is totally bounded under the weaker assumption

<sup>1</sup>The diameter convention is used throughout. Replacing it by a comparable covering convention does not change box dimension, but can change exact covering numbers and normalized covering constants.

that every  $M(T)$  is finite: at any scale  $\varepsilon > 0$ , cover  $M(2/\varepsilon)^{-1}$  point by point and put the remaining reciprocals in one set of diameter at most  $\varepsilon$ .

**The main theorem.** Set

$$\beta := \frac{\alpha d}{\alpha + d}.$$

We call  $\beta$  the *harmonic value*, since

$$\frac{1}{\beta} = \frac{1}{\alpha} + \frac{1}{d}.$$

For  $\theta > 0$  and  $T \geq 2$ , let  $P(\theta, T)$  denote the assertion that there is a set  $S_T \subset M(T)$  such that

$$\#S_T \geq T^{\alpha-\theta} \quad \text{and} \quad |x - y| \geq T^{1-\alpha/d-\theta} \quad \text{for all } x \neq y \in S_T.$$

We say that  $M$  has *subsequential critical-scale packing* if, for every  $\theta > 0$ ,  $P(\theta, T)$  holds for arbitrarily large  $T$ . We say that  $M$  has *all-scale critical-scale packing* if, for every  $\theta > 0$ ,  $P(\theta, T)$  holds for every sufficiently large  $T$ .

**Theorem 1.1** (Harmonic upper bound and packing criterion). *Let  $(K, |\cdot|)$  have ball-covering dimension  $d \in (0, \infty)$ , and let  $M \subset K^\times$  satisfy  $|x| \geq 1$  and have counting exponent  $\alpha > 0$ . Then*

$$\overline{\dim}_B(M^{-1}) \leq \beta = \frac{\alpha d}{\alpha + d}.$$

Moreover:

- (i)  $\overline{\dim}_B(M^{-1}) = \beta$  if and only if  $M$  has subsequential critical-scale packing, and
- (ii)  $\dim_B(M^{-1}) = \beta$  if and only if  $M$  has all-scale critical-scale packing.

The upper bound in Theorem 1.1 comes from a head–tail decomposition. For every  $s > d$ ,

$$(4) \quad N(M^{-1}, \varepsilon) \leq A_M(T) + C_s(\varepsilon T)^{-s} \quad (T \geq 1, 0 < \varepsilon \leq T^{-1}).$$

The head consists of at most  $A_M(T)$  points, while the tail lies in  $B(0, T^{-1})$ . Balancing the two terms gives  $\overline{\dim}_B(M^{-1}) \leq \alpha s/(\alpha + s)$ ; letting  $s \downarrow d$  gives the stated bound.

The same balance identifies the relevant separation scale. Distributing  $T^\alpha$  points throughout a region of linear scale  $T$  in dimension  $d$  suggests spacing  $T^{1-\alpha/d}$ . The identity

$$(5) \quad |x^{-1} - y^{-1}| = \frac{|x - y|}{|x| |y|}$$

then gives reciprocal spacing at least  $T^{-1-\alpha/d}$  for such a packing in  $M(T)$ . At that scale,  $T^\alpha$  separated reciprocals give the exponent

$$\frac{\alpha}{1 + \alpha/d} = \beta.$$

The converse in Theorem 1.1 shows that saturation of the upper bound forces such packings, with arbitrarily small losses in the exponents.

**Criteria, examples, and arithmetic applications.** In Theorem 2.3, we give an annular non-concentration criterion for all-scale critical-scale packing. Uniform discreteness gives the complementary lower bound  $\underline{\dim}_B(M^{-1}) \geq \alpha/2$ , and hence reciprocal dimension  $d/2$  when  $\alpha = d$  (Proposition 3.1). For every  $0 < \alpha < 1$ , sparse integer examples attain both  $\alpha/2$  and  $\alpha/(1 + \alpha)$ , showing that counting growth alone does not determine reciprocal dimension.

We apply these criteria to power sequences, monic polynomials, and Gaussian integers. For monic polynomials, we also compute exact covering numbers and exhibit nonconvergence of the normalized covering number. Further arithmetic applications give reciprocal dimension  $1/2$  for distinct ideal norms and two-sided covering estimates for balanced algebraic integers and definite quaternion orders.

Finally, we observed that positive counting exponent or positive upper reciprocal box dimension obstructs containment in a monoid generated by finitely many expanding elements with finitely many absolute-value-1 factors. This allows us to extend the Saito-style multiplicative argument motivating the paper.

**Related work.** The geometrically-closest context is box dimension at infinity. Radunović, Žubrinić, and Županović [RŽŽ12] study unbounded sets through geometric inversion and discuss reciprocal power sequences. In  $\mathbb{R}$ , geometric inversion is field inversion; in  $\mathbb{C}$ , the two differ by complex conjugation, an isometry. For reciprocal Gaussian integers, Gu and Jiang [GJ26] prove dimension 1 and compute the critical Minkowski content. (Their content formula concerns the area of small neighborhoods, rather than minimum covering numbers.)

Also related is the dimension theory of discrete sets. Iosevich, Rudnev, and Uriarte-Tuero [IRUT14] develop discrete Minkowski and Hausdorff dimensions for families of large finite separated sets. Saito [Sai18, Theorem 1.1] identifies upper zeta dimension with the exponent of convergence of the norm Dirichlet series for uniformly discrete Euclidean sets. When a counting exponent exists, it equals this exponent of convergence. These notions describe the large-scale growth and distribution of denominators, whereas reciprocal box dimension describes their small-scale geometry after inversion.

Our contribution is the harmonic upper bound linking denominator growth to reciprocal geometry, together with the packing characterization of equality.

**Road map.** Section 2 proves the main theorem and the annular criterion. Section 3 gives the basic examples, the exact polynomial covering formula, and the sharp sparse constructions. Section 4 treats ideal norms, balanced

algebraic integers, and definite quaternion orders. Section 5 gives finite-generation consequences, records the zeta interpretation of the counting exponent, and formulates an open packing problem.

## 2. COUNTING AND PACKING UNDER INVERSION

**2.1. Conventions and standing hypotheses.** Throughout,  $\mathbb{N} := \{1, 2, \dots\}$ . Subscripts on  $O$ ,  $\ll$ ,  $\gg$ , and  $\asymp$  indicate permitted dependence; the notation  $X \asymp Y$  means both  $X \ll Y$  and  $Y \ll X$ . Each  $o(1)$  refers to the limiting variable indicated in its context.

Throughout this section,  $(K, |\cdot|)$  has ball-covering dimension  $d \in (0, \infty)$ , and  $M \subset K^\times$  satisfies  $|x| \geq 1$  and (2) for some  $\alpha > 0$ . Every  $M(T)$  is finite, and  $M^{-1}$  is totally bounded. For every  $s > d$ , the estimate (3) is available.

**2.2. The harmonic upper bound.** We first record the covering estimate underlying the main theorem.

**Lemma 2.1** (Head–tail estimate). *For every admissible ball-covering exponent  $s$ ,*

$$N(M^{-1}, \varepsilon) \leq A_M(T) + C_s(\varepsilon T)^{-s} \quad (T \geq 1, 0 < \varepsilon \leq T^{-1}).$$

*Proof.* The reciprocals of the elements of  $M(T)$  can be covered individually. Every remaining reciprocal lies in  $B(0, T^{-1})$ , whose covering number is at most  $C_s(\varepsilon T)^{-s}$  by (3).  $\square$

*Proof of the upper bound in Theorem 1.1.* For  $s > d$ , take

$$T = \varepsilon^{-s/(\alpha+s)}.$$

Then  $\varepsilon < T^{-1}$  for  $0 < \varepsilon < 1$ , and Lemma 2.1 gives

$$N(M^{-1}, \varepsilon) \leq \varepsilon^{-\alpha s/(\alpha+s)+o(1)}.$$

Consequently,

$$\overline{\dim}_B(M^{-1}) \leq \frac{\alpha s}{\alpha + s}.$$

Letting  $s \downarrow d$  proves

$$(6) \quad \overline{\dim}_B(M^{-1}) \leq \beta := \frac{\alpha d}{\alpha + d}. \quad \square$$

**2.3. Packing and saturation.** We now prove the equality statements in Theorem 1.1. Put

$$\gamma := 1 + \frac{\alpha}{d}, \quad \beta\gamma = \alpha.$$

*Proof of the packing statements in Theorem 1.1. Sufficiency.* Suppose that  $P(\theta, T)$  holds, witnessed by  $S_T \subset M(T)$ . For distinct  $x, y \in S_T$ ,

$$|x^{-1} - y^{-1}| = \frac{|x - y|}{|x||y|} \geq T^{-1-\alpha/d-\theta} = T^{-\gamma-\theta}.$$

Thus, at scale

$$\varepsilon_T := \frac{1}{2}T^{-\gamma-\theta},$$

each covering set contains at most one point of  $S_T^{-1}$ , so

$$(7) \quad N(M^{-1}, \varepsilon_T) \geq T^{\alpha-\theta}.$$

If  $M$  has subsequential critical-scale packing, then for each fixed sufficiently small  $\theta > 0$ , this inequality holds along arbitrarily large  $T$ . Hence

$$\overline{\dim}_B(M^{-1}) \geq \frac{\alpha - \theta}{\gamma + \theta}.$$

Letting  $\theta \downarrow 0$  and using (6) proves  $\overline{\dim}_B(M^{-1}) = \beta$ .

If  $M$  has all-scale critical-scale packing, fix a sufficiently small  $\theta > 0$  and, for any sufficiently small  $\varepsilon > 0$ , take

$$T = (2\varepsilon)^{-1/(\gamma+\theta)}.$$

Then  $P(\theta, T)$  holds and  $\varepsilon = \frac{1}{2}T^{-\gamma-\theta}$ . Applying (7) at every sufficiently small  $\varepsilon$  gives

$$\underline{\dim}_B(M^{-1}) \geq \frac{\alpha - \theta}{\gamma + \theta}.$$

Letting  $\theta \downarrow 0$  proves  $\dim_B(M^{-1}) = \beta$ .

*Necessity for upper box dimension.* Suppose that  $\overline{\dim}_B(M^{-1}) = \beta$ . There is a sequence  $\varepsilon \downarrow 0$  along which

$$(8) \quad N(M^{-1}, 2\varepsilon) = \varepsilon^{-\beta+o(1)}.$$

Write  $\varepsilon = T^{-\gamma}$ , so  $T \rightarrow \infty$  along the same sequence.

Fix  $\theta > 0$ . Choose  $0 < \delta < \min\{1, \alpha/d\}$  sufficiently small that

$$(9) \quad \frac{\alpha}{1+\delta} > \alpha - \theta, \quad \frac{1 - \alpha/d - 2\delta}{1+\delta} > 1 - \frac{\alpha}{d} - \theta.$$

Both choices are possible by continuity at  $\delta = 0$ . Next choose  $s > d$  sufficiently close to  $d$  that

$$s(\alpha/d - \delta) < \alpha, \quad \kappa := \alpha - s(\alpha/d - \delta) > 0.$$

Split the denominators into three ranges:

$$|x| < T^{1-\delta}, \quad T^{1-\delta} \leq |x| \leq T^{1+\delta}, \quad |x| > T^{1+\delta}.$$

The first range contributes at most

$$A_M(T^{1-\delta}) = T^{\alpha(1-\delta)+o(1)}$$

reciprocal points. The reciprocals from the third range lie in  $B(0, T^{-1-\delta})$ . Since  $\delta < \alpha/d$ , for all large  $T$  we have  $2\varepsilon \leq T^{-1-\delta}$ , and (3) gives

$$N(B(0, T^{-1-\delta}), 2\varepsilon) \ll_s T^{s(\alpha/d-\delta)} = T^{\alpha-\kappa}.$$

On the other hand, (8) gives

$$N(M^{-1}, 2\varepsilon) = T^{\alpha+o(1)}.$$

Thus neither the small-denominator range nor the far tail can account for the leading covering number. More precisely, if

$$c := \frac{1}{2} \min\{\alpha\delta, \kappa\} > 0,$$

then each of those two ranges can be covered using  $O(T^{\alpha-c})$  sets of diameter at most  $2\varepsilon$ . Subadditivity of covering numbers therefore implies that the finite middle reciprocal set

$$E_T := \{x^{-1} : x \in M, T^{1-\delta} \leq |x| \leq T^{1+\delta}\}$$

satisfies

$$(10) \quad N(E_T, 2\varepsilon) \geq T^{\alpha-o(1)}.$$

Choose an inclusion-maximal subset  $F_T \subset E_T$  whose distinct points are more than  $\varepsilon$  apart. The closed  $\varepsilon$ -balls centered at  $F_T$  cover  $E_T$ , and their diameters are at most  $2\varepsilon$ . Hence

$$\#F_T \geq N(E_T, 2\varepsilon) \geq T^{\alpha-o(1)}.$$

Let  $S_T$  be the corresponding set of denominators. For distinct  $x, y \in S_T$ ,

$$|x - y| = |x| |y| |x^{-1} - y^{-1}| > T^{2(1-\delta)} \varepsilon = T^{1-\alpha/d-2\delta}.$$

Set  $R = T^{1+\delta}$ . Then  $S_T \subset M(R)$  and

$$\#S_T \geq R^{\alpha/(1+\delta)-o(1)}, \quad |x - y| > R^{(1-\alpha/d-2\delta)/(1+\delta)}.$$

The strict inequalities in (9) absorb the subpolynomial loss and give, for all sufficiently large  $T$  along the chosen sequence,

$$\#S_T \geq R^{\alpha-\theta}, \quad |x - y| \geq R^{1-\alpha/d-\theta}.$$

Thus  $P(\theta, R)$  holds for arbitrarily large  $R$ .

*Necessity for box dimension.* If  $\dim_B(M^{-1}) = \beta$ , then (8) holds as  $\varepsilon \downarrow 0$  without restriction to a subsequence. For fixed  $\theta$ , choose  $\delta$  and  $s$  as above. The construction then works for every sufficiently large  $T$ . Since  $R = T^{1+\delta}$  ranges over every sufficiently large positive real number,  $P(\theta, R)$  holds at every sufficiently large height. This is all-scale critical-scale packing.  $\square$

*Remark 2.2* (Division algebras). Theorem 1.1 also holds for a possibly non-commutative division algebra  $D$  equipped with an absolute value satisfying the triangle inequality and  $|xy| = |x| |y|$ . Define its ball-covering dimension using the metric  $|x - y|$ , exactly as above. The identity

$$x^{-1} - y^{-1} = x^{-1}(y - x)y^{-1}$$

gives

$$|x^{-1} - y^{-1}| = \frac{|x - y|}{|x| |y|},$$

which is the only algebraic property of inversion used in the proof. The finite-generation discussion in Section 5, however, is stated for commutative fields.

**2.4. An annular non-concentration criterion.** Our packing conditions are intrinsic, but applications often provide local counting estimates instead. The following annular estimate is sufficient for our all-scale packing criterion.

**Theorem 2.3** (Annular criterion). *In the setting of Theorem 1.1, suppose that for every  $\eta > 0$  there is a constant  $C_\eta$  such that*

$$(11) \quad \#\{x \in M : t \leq |x| < 2t, |x - z| \leq t^{1-\alpha/d}\} \leq C_\eta t^\eta$$

*for all  $t \geq 2$  and  $z \in K$ . Then  $M$  has all-scale critical-scale packing. In particular,*

$$\dim_B(M^{-1}) = \frac{\alpha d}{\alpha + d}.$$

*Proof.* Fix  $0 < \delta < 1$ , put  $h := 1 - \alpha/d$ , and define

$$F_T := M(T) \setminus M(T^{1-\delta}), \quad r_T := T^{h-\delta}.$$

The counting law implies

$$(12) \quad \#F_T = A_M(T) - A_M(T^{1-\delta}) = T^{\alpha+o(1)}.$$

Cover the height range  $T^{1-\delta} < |x| \leq T$  by  $O(\log T)$  dyadic annuli  $t \leq |x| < 2t$ , where  $T^{1-\delta}/2 \leq t \leq T$ . For all sufficiently large  $T$ , we have  $r_T \leq t^h$  throughout this range. Indeed, if  $h \geq 0$ , then

$$\frac{r_T}{t^h} \leq 2^h T^{-\delta\alpha/d} \rightarrow 0;$$

if  $h < 0$ , then  $t^h \geq T^h > r_T$ .

Applying (11) in these annuli gives

$$Q_T := \sup_{z \in K} \#(F_T \cap B(z, r_T)) \leq T^{o(1)}.$$

To see the subpolynomial bound explicitly, for each fixed  $\eta > 0$  the annular estimate gives  $Q_T \ll_\eta T^\eta \log T$ .

Select a point of  $F_T$ , discard its closed  $r_T$ -ball, and repeat until no points remain. At each step at most  $Q_T$  points are discarded. The selected set has pairwise distances greater than  $r_T$  and cardinality at least

$$\frac{\#F_T}{Q_T} \geq T^{\alpha-o(1)}.$$

Given  $\theta > 0$ , choose  $0 < \delta < \min\{1, \theta\}$  and take  $T$  sufficiently large to absorb the subpolynomial loss. The selected set then verifies  $P(\theta, T)$ . Since the construction works at every sufficiently large  $T$ , the result follows from Theorem 1.1.  $\square$

*Remark 2.4* (Why a cumulative condition can be too strong). A superficially similar cumulative estimate would be

$$(13) \quad \#(M(T) \cap B(z, \rho)) \ll_\eta T^\eta \left(1 + T^\alpha(\rho/T)^d\right) \quad (0 < \rho \leq T),$$

uniformly in  $z$ , for every  $\eta > 0$ .

Under the counting law, this condition excludes every  $0 < \alpha < d$ . Indeed, set  $z = 0$  and  $\rho = T^{1-\alpha/d}$ . The left-hand side is

$$A_M(\rho) = T^{\alpha(1-\alpha/d)+o(1)},$$

whereas the right-hand side is  $O_\eta(T^\eta)$ . Choosing  $\eta < \alpha(1-\alpha/d)$  gives a contradiction. The annular condition avoids this accumulation of denominators from earlier height scales.

### 3. EXAMPLES AND SHARPNESS

**3.1. Power sequences.** Let

$$M_a := \{n^a : n \geq 1\} \subset \mathbb{R} \quad (a > 0).$$

Its counting function is  $A_{M_a}(T) = \lfloor T^{1/a} \rfloor$ , so its counting exponent is  $\alpha = 1/a$ . On the annulus  $t \leq n^a < 2t$ , the mean-value theorem gives

$$(n+1)^a - n^a \asymp_a t^{1-1/a}.$$

Consequently, every interval of radius  $t^{1-1/a}$  meets the annular set in  $O_a(1)$  points. Applying Theorem 2.3 with  $d = 1$  yields

$$(14) \quad \dim_B \{n^{-a} : n \geq 1\} = \frac{1}{a+1}.$$

This classical example also appears in the literature on inversion and fractal strings; see [RŽŽ12, LvF13].

**3.2. Uniformly discrete denominator sets.** A subset of a metric space is *uniformly discrete* if the distances between distinct points have a positive lower bound.

**Proposition 3.1** (Uniformly discrete sets). *Let  $(K, |\cdot|)$  have ball-covering dimension  $d \in (0, \infty)$ , and let  $M \subset K^\times$  satisfy  $|x| \geq 1$  and have counting exponent  $\alpha > 0$ . If  $M$  is uniformly discrete, then*

$$\frac{\alpha}{2} \leq \underline{\dim}_B(M^{-1}) \leq \overline{\dim}_B(M^{-1}) \leq \frac{\alpha d}{\alpha + d}.$$

*In particular,  $\alpha \leq d$ . If  $\alpha = d$ , then*

$$\dim_B(M^{-1}) = \frac{d}{2}.$$

*Proof.* The upper bound is Theorem 1.1. Choose  $\rho_0 > 0$  such that

$$|x - y| \geq \rho_0 \quad (x \neq y \text{ in } M).$$

For small  $\varepsilon > 0$ , take

$$T = \left(\frac{\rho_0}{2\varepsilon}\right)^{1/2}.$$

If  $x, y \in M(T)$  are distinct, then

$$|x^{-1} - y^{-1}| = \frac{|x - y|}{|x| |y|} \geq \rho_0 T^{-2} = 2\varepsilon.$$

Therefore

$$N(M^{-1}, \varepsilon) \geq A_M(T) = \varepsilon^{-\alpha/2+o(1)},$$

which proves the lower bound. Comparing the lower and upper bounds gives  $\alpha \leq d$ ; when  $\alpha = d$ , they coincide.  $\square$

Under Proposition 3.1, uniform discreteness of  $\mathbb{N} \subset \mathbb{R}$  recovers (1), since in that case  $\alpha = d = 1$ .

For a prime power  $q$ , equip  $K = \mathbb{F}_q((1/t))$  with the absolute value determined by  $|t| = q$ . Let

$$M_q := \{f \in \mathbb{F}_q[t] : f \text{ is monic and nonzero}\}.$$

This includes the constant polynomial 1. Distinct polynomials are at distance at least 1. If  $q^\ell \leq T < q^{\ell+1}$ , with  $\ell \in \mathbb{Z}_{\geq 0}$ , then

$$A_{M_q}(T) = 1 + q + \cdots + q^\ell \asymp_q T.$$

Since  $d = 1$ , Proposition 3.1 gives

$$(15) \quad \dim_B\{1/f : f \in M_q\} = \frac{1}{2}.$$

Similarly, the Gaussian integers form a uniformly discrete lattice in  $\mathbb{C}$ , and

$$\#\{z \in \mathbb{Z}[i] \setminus \{0\} : |z| \leq T\} = \pi T^2 + O(T).$$

Here  $\alpha = d = 2$ , so

$$(16) \quad \dim_B\{1/z : 0 \neq z \in \mathbb{Z}[i]\} = 1.$$

Removing the four reciprocal units does not change the box dimension and gives the set  $G_1$  studied by Gu and Jiang [GJ26], recovering their first-level dimension formula  $\dim_B G_1 = 1$ .

**3.3. An exact polynomial covering formula.** Retain  $K = \mathbb{F}_q((1/t))$  and  $M_q$  from the preceding subsection. Put

$$B_q := M_q^{-1}, \quad S_j := \{1/f : f \in M_q, \deg f = j\} \quad (j \in \mathbb{Z}_{\geq 0}).$$

We call  $S_j$  the *degree- $j$  shell*.

**Proposition 3.2** (Exact polynomial covering numbers). *For every integer  $r \geq 1$ ,*

$$(17) \quad N(B_q, q^{-r}) = 1 + \sum_{j=0}^{r-1} q^{\min\{j, r-j-1\}}.$$

Consequently,

$$(18) \quad N(B_q, q^{-2m}) = 1 + \frac{2(q^m - 1)}{q - 1} \quad (m \geq 1),$$

$$(19) \quad N(B_q, q^{-(2m+1)}) = 1 + \frac{2(q^m - 1)}{q - 1} + q^m \quad (m \geq 0).$$

In particular,

$$\lim_{m \rightarrow \infty} q^{-m} N(B_q, q^{-2m}) = \frac{2}{q-1},$$

$$\lim_{m \rightarrow \infty} q^{-m-1/2} N(B_q, q^{-(2m+1)}) = \frac{q+1}{(q-1)\sqrt{q}}.$$

Thus  $\varepsilon^{1/2} N(B_q, \varepsilon)$  does not converge as  $\varepsilon \downarrow 0$ .

*Proof.* In an ultrametric space, the relation  $|x - y| \leq q^{-r}$  is an equivalence relation. Its classes have diameter at most  $q^{-r}$ , and every set of diameter at most  $q^{-r}$  is contained in one class. Hence  $N(B_q, q^{-r})$  is exactly the number of classes meeting  $B_q$ .

The tail  $\bigcup_{j \geq r} S_j$  lies in  $B(0, q^{-r})$  and contributes one class. Points from two different shells  $S_j, S_k$  have distance  $q^{-\min\{j, k\}}$ . Thus no class meets two distinct shells with degrees less than  $r$ , or both such a shell and the tail.

For distinct monic polynomials  $f, g$  of degree  $j < r$ ,

$$\left| \frac{1}{f} - \frac{1}{g} \right| = q^{\deg(f-g)-2j}.$$

They belong to the same class precisely when  $\deg(f - g) \leq 2j - r$ . If  $2j - r < 0$ , each of the  $q^j$  polynomials gives a different class. If  $2j - r \geq 0$ , the coefficients in degrees  $2j - r + 1, \dots, j - 1$  determine the class, giving  $q^{r-j-1}$  classes. Therefore the shell contributes  $q^{\min\{j, r-j-1\}}$  classes, proving (17).

For  $r = 2m$ , the exponents in the sum are

$$0, 1, \dots, m-1, m-1, \dots, 1, 0.$$

For  $r = 2m + 1$ , the central exponent  $m$  occurs once and each smaller exponent occurs twice. Summing gives (18) and (19), and hence the two limits. They are distinct because  $q + 1 > 2\sqrt{q}$ .  $\square$

### 3.4. Sharpness for sparse integer sets.

**Proposition 3.3** (Sharp sparse bounds). *Let  $M \subset \mathbb{N}$  satisfy*

$$\#(M \cap [1, T]) = T^{\alpha+o(1)}$$

*for some  $0 < \alpha < 1$ . Then*

$$(20) \quad \frac{\alpha}{2} \leq \underline{\dim}_B(M^{-1}) \leq \overline{\dim}_B(M^{-1}) \leq \frac{\alpha}{1+\alpha}.$$

*For every  $0 < \alpha < 1$ , both endpoints are attained by sets whose reciprocal box dimensions exist.*

*Proof.* The inequalities follow from Proposition 3.1 with  $d = 1$ .

For the upper endpoint, define

$$M_\alpha^+ := \{\lfloor n^{1/\alpha} \rfloor : n \geq 1\}.$$

Since  $1/\alpha > 1$ , this sequence is strictly increasing. Its counting function satisfies

$$\#(M_\alpha^+ \cap [1, T]) = T^\alpha + O_\alpha(1).$$

Indeed,  $\lfloor n^{1/\alpha} \rfloor \leq T$  is equivalent to  $n^{1/\alpha} < \lfloor T \rfloor + 1$ .

Consecutive terms in an annulus  $[t, 2t)$  are separated by  $\asymp_\alpha t^{1-\alpha}$  for all sufficiently large  $t$ . The corresponding gaps before taking floors have that size and tend to infinity, while the floor error is bounded. The annular criterion therefore gives

$$\dim_B((M_\alpha^+)^{-1}) = \frac{\alpha}{1 + \alpha}.$$

For the lower endpoint, define

$$M_\alpha^- := \bigcup_{k \geq 1} \{2^k + j : 0 \leq j < \lfloor 2^{\alpha k} \rfloor\}.$$

The blocks are disjoint. Summing their cardinalities gives

$$(21) \quad \#(M_\alpha^- \cap [1, T]) \asymp_\alpha T^\alpha.$$

Thus (20) gives

$$\underline{\dim}_B((M_\alpha^-)^{-1}) \geq \frac{\alpha}{2}.$$

The reciprocals from block  $k$  lie in an interval of length  $O(2^{-(2-\alpha)k})$ . Given small  $\varepsilon > 0$ , set

$$J := \left\lfloor \frac{1}{2} \log_2(1/\varepsilon) \right\rfloor, \quad J_+ := \lceil \log_2(1/\varepsilon) \rceil.$$

Cover all blocks  $k \leq J$  point by point. This costs  $O_\alpha(\varepsilon^{-\alpha/2})$  sets. For  $J < k \leq J_+$ , cover the interval containing the reciprocal block using

$$O\left(1 + \varepsilon^{-1} 2^{-(2-\alpha)k}\right)$$

sets. Summing gives

$$O_\alpha\left(\log(1/\varepsilon) + \varepsilon^{-\alpha/2}\right) = O_\alpha(\varepsilon^{-\alpha/2}).$$

All blocks with  $k > J_+$  lie in a single interval of length at most  $\varepsilon$ . Therefore

$$N((M_\alpha^-)^{-1}, \varepsilon) \ll_\alpha \varepsilon^{-\alpha/2},$$

and

$$\dim_B((M_\alpha^-)^{-1}) = \frac{\alpha}{2}.$$

□

The two constructions in the proof of Proposition 3.3 have the same counting exponent and are both uniformly discrete. Their different reciprocal dimensions show why a distributional hypothesis is needed in addition to counting growth. Theorem 1.1 identifies the relevant distinction as critical-scale packing.

## 4. NUMBER-FIELD AND QUATERNIONIC APPLICATIONS

We now give three applications. Distinct ideal norms are handled by Proposition 3.1; balanced algebraic integers require a coordinatewise lattice argument; and definite quaternion orders use the division-algebra extension of inversion.

Throughout the first two subsections, let  $L$  be a number field of signature  $(r_1, r_2)$  and degree  $n = r_1 + 2r_2$ , with ring of integers  $\mathcal{O}_L$ . Choose one embedding for each Archimedean place and write

$$\sigma : L \longrightarrow V := \mathbb{R}^{r_1} \times \mathbb{C}^{r_2}, \quad \sigma(a) = (\sigma_v(a))_{v|\infty}.$$

Equip  $V$  with the sup norm, using ordinary modulus in each complex coordinate. The image  $\sigma(\mathcal{O}_L)$  is a full lattice in the real  $n$ -dimensional vector space  $V$ .

We use the elementary lattice-counting fact that, for a full lattice  $\Lambda \subset \mathbb{R}^n$  and a bounded Jordan-measurable set  $D$ ,

$$(22) \quad \#(\Lambda \cap TD) = \frac{\text{vol}(D)}{\text{covol}(\Lambda)} T^n + o(T^n);$$

this follows by comparison with translates of a fundamental parallelepiped, or equivalently via Riemann sums after a linear change of coordinates.

Write  $N_{L/\mathbb{Q}}$  for the field norm and

$$N \mathfrak{a} := [\mathcal{O}_L : \mathfrak{a}]$$

for the absolute norm of a nonzero integral ideal  $\mathfrak{a}$ .

**4.1. Distinct ideal norms.** Define

$$S_L^{\text{pr}} := \{|N_{L/\mathbb{Q}}(a)| : 0 \neq a \in \mathcal{O}_L\}$$

and

$$S_L := \{N \mathfrak{a} : 0 \neq \mathfrak{a} \subseteq \mathcal{O}_L\};$$

these are the sets of distinct norms of nonzero principal integral ideals and of all nonzero integral ideals, respectively. We regard both as subsets of  $\mathbb{R}$  with its usual metric.

**Theorem 4.1** (Distinct ideal norms). *As  $X \rightarrow \infty$ ,*

$$\#(S_L^{\text{pr}} \cap [1, X]) = X^{1+o(1)}, \quad \#(S_L \cap [1, X]) = X^{1+o(1)}.$$

*Consequently,*

$$\dim_B((S_L^{\text{pr}})^{-1}) = \dim_B(S_L^{-1}) = \frac{1}{2}.$$

*Proof.* Consider the Archimedean region

$$(23) \quad T \leq |x_v| \leq 2T \quad (v \mid \infty).$$

It is the dilation by  $T$  of a bounded Jordan-measurable set of positive volume. By (22), it contains  $\asymp_L T^n$  points of  $\sigma(\mathcal{O}_L)$ . Every corresponding algebraic integer  $a$  satisfies

$$|N_{L/\mathbb{Q}}(a)| \leq (2T)^n,$$

where complex places contribute with multiplicity two.

Only  $\mathcal{O}_L(1)$  of these algebraic integers can generate any one principal ideal. Indeed, if  $(a) = (b)$ , then  $a = ub$  for a unit  $u$ , and (23) gives

$$\frac{1}{2} \leq |\sigma_v(u)| \leq 2 \quad (v \mid \infty).$$

There are only finitely many such units: their Minkowski images are lattice points in a fixed bounded region. Taking  $T = X^{1/n}/2$  therefore produces  $\gg_L X$  principal integral ideals of norm at most  $X$ .

To pass from ideals to distinct norms, put

$$a_L(m) := \#\{0 \neq \mathfrak{a} \subseteq \mathcal{O}_L : N \mathfrak{a} = m\}.$$

For a rational prime  $p$ , write

$$p\mathcal{O}_L = \mathfrak{p}_1^{e_1} \cdots \mathfrak{p}_g^{e_g}, \quad N \mathfrak{p}_j = p^{f_j}.$$

The ideals supported above  $p$  and having norm  $p^\nu$  correspond to nonnegative solutions of

$$\sum_{j=1}^g f_j u_j = \nu.$$

There are at most  $(\nu+1)^g \leq (\nu+1)^n$  such solutions. By unique factorization of ideals,

$$a_L(m) \leq \tau(m)^n,$$

where  $\tau$  is the divisor function. The elementary estimate  $\tau(m) \ll_\eta m^\eta$  for every  $\eta > 0$  implies

$$(24) \quad \max_{m \leq X} a_L(m) \leq X^{o(1)}.$$

Dividing the  $\gg_L X$  principal ideals constructed above by this maximal multiplicity yields

$$\#(S_L^{\text{pr}} \cap [1, X]) \geq X^{1-o(1)}.$$

The reverse bound follows from  $S_L^{\text{pr}} \subset \mathbb{N}$ . Since

$$S_L^{\text{pr}} \subset S_L \subset \mathbb{N},$$

the same exponent holds for  $S_L$ .

Both norm sets are uniformly discrete subsets of  $\mathbb{N}$  with counting exponent 1. Applying Proposition 3.1 with  $d = 1$  proves the reciprocal dimension statements.  $\square$

**4.2. Balanced algebraic integers.** We now study  $\sigma(a^{-1})$  itself rather than the reciprocal of an integer-valued norm. The map  $\sigma(a) \mapsto \sigma(a^{-1})$  is coordinatewise inversion in  $V$ . The sup norm on  $V$  is not a multiplicative absolute value on  $L$ , so this is a lattice analogue of Theorem 1.1, not a direct application.

Without a restriction, the reciprocal set can be unbounded. For example, in a real quadratic field, powers of a suitable infinite-order unit have coordinatewise absolute values  $(\lambda^k, \lambda^{-k})$  for some  $\lambda > 1$ .

For  $a \in L$ , define the Archimedean sup height

$$H(a) := \max_{v|\infty} |\sigma_v(a)|.$$

Fix  $0 < c < 1$  and set

$$M_c := \{0 \neq a \in \mathcal{O}_L : \min_{v|\infty} |\sigma_v(a)| \geq cH(a)\}, \quad E_c := \{\sigma(a^{-1}) : a \in M_c\}.$$

The balance condition prevents any coordinate from being small relative to the height.

**Theorem 4.2** (Balanced lattice reciprocals). *With the sup-norm metric on  $V$ , we have*

$$(25) \quad N(E_c, \varepsilon) \asymp_{L,c} \varepsilon^{-n/2} \quad (\varepsilon \downarrow 0).$$

In particular,  $\dim_B E_c = n/2$ .

*Proof.* For every nonzero algebraic integer,

$$1 \leq |N_{L/\mathbb{Q}}(a)| \leq H(a)^n,$$

so  $H(a) \geq 1$ . If  $a \in M_c$ , then

$$\|\sigma(a^{-1})\|_\infty = \frac{1}{\min_v |\sigma_v(a)|} \leq \frac{1}{cH(a)} \leq \frac{1}{c}.$$

Thus  $E_c$  is bounded.

The set

$$D_c := \{x \in V : \|x\|_\infty \leq 1, \min_v |x_v| \geq c\|x\|_\infty\}$$

is bounded and Jordan-measurable, with nonempty interior. By (22),

$$(26) \quad \#\{a \in M_c : H(a) \leq T\} \asymp_{L,c} T^n.$$

Choose  $s_L > 0$  such that

$$\|\sigma(a) - \sigma(b)\|_\infty \geq s_L \quad (a \neq b \text{ in } \mathcal{O}_L).$$

For distinct  $a, b \in M_c$  with  $H(a), H(b) \leq T$ , choose a coordinate  $v$  such that  $|\sigma_v(a - b)| \geq s_L$ . Then

$$\|\sigma(a^{-1}) - \sigma(b^{-1})\|_\infty \geq \frac{|\sigma_v(a - b)|}{|\sigma_v(a)| |\sigma_v(b)|} \geq \frac{s_L}{T^2}.$$

Taking  $T = (s_L/(2\varepsilon))^{1/2}$  and using (26) proves the lower bound in (25).

For the upper bound, cover all reciprocals with  $H(a) \leq T$  individually, at a cost of  $O_{L,c}(T^n)$  sets. The remaining reciprocals lie in the sup-norm ball of radius  $1/(cT)$  centered at 0. A sup-norm ball in  $V$  of radius  $R$  can be covered at scale  $\varepsilon \leq R$  by  $O_n((R/\varepsilon)^n)$  sets. Thus the tail costs  $O_{L,c}((\varepsilon T)^{-n})$  sets. Taking  $T = \varepsilon^{-1/2}$  gives the matching upper bound.  $\square$

The same proof applies to any order  $\mathcal{O} \subset L$ , with the implied constants allowed to depend on  $\mathcal{O}$  and  $c$ .

**4.3. Definite quaternion orders.** Let  $A$  be a definite quaternion algebra over  $\mathbb{Q}$ . Identify  $A \otimes_{\mathbb{Q}} \mathbb{R}$  with Hamilton's quaternion division algebra  $\mathbb{H} \simeq \mathbb{R}^4$ . Write  $\text{nrd}$  for the reduced norm and equip  $\mathbb{H}$  with the multiplicative Euclidean norm

$$|x| := \sqrt{\text{nrd}(x)}.$$

All covering numbers in this subsection use its induced metric.

**Proposition 4.3** (Definite quaternion orders). *Let  $\mathcal{O}$  be an order in  $A$ . Then*

$$(27) \quad N(\{x^{-1} : 0 \neq x \in \mathcal{O}\}, \varepsilon) \asymp_{\mathcal{O}} \varepsilon^{-2} \quad (\varepsilon \downarrow 0).$$

*In particular, the reciprocal set has box dimension 2.*

*Proof.* The order  $\mathcal{O}$  is a rank-4 lattice in  $\mathbb{H}$ . For  $0 \neq x \in \mathcal{O}$ , the reduced norm is a positive integer, so  $|x| \geq 1$ . Lattice counting gives

$$\#\{0 \neq x \in \mathcal{O} : |x| \leq T\} \asymp_{\mathcal{O}} T^4.$$

Moreover, by Remark 2.2,

$$|x^{-1} - y^{-1}| = \frac{|x - y|}{|x||y|}.$$

For the upper bound, split at  $T = \varepsilon^{-1/2}$ . The head costs  $O_{\mathcal{O}}(T^4)$  sets. The tail lies in the Euclidean ball of radius  $T^{-1}$  centered at 0, which costs  $O((\varepsilon T)^{-4})$  sets.

For the lower bound, choose  $s_{\mathcal{O}} > 0$  such that

$$|x - y| \geq s_{\mathcal{O}} \quad (x \neq y \text{ in } \mathcal{O}).$$

Take  $T = (s_{\mathcal{O}}/(2\varepsilon))^{1/2}$ . Distinct nonzero  $x, y \in \mathcal{O}$  of norm at most  $T$  have reciprocals separated by at least  $2\varepsilon$ . Therefore

$$N(\{x^{-1} : 0 \neq x \in \mathcal{O}\}, \varepsilon) \gg_{\mathcal{O}} T^4 \asymp_{\mathcal{O}} \varepsilon^{-2}. \quad \square$$

## 5. FINITE-GENERATION CONSEQUENCES AND FURTHER QUESTIONS

**5.1. Finite generation.** We now separate the multiplicative input in Saito's argument from the reciprocal dimension calculation.

Let  $K$  be a field equipped with a multiplicative absolute value satisfying the triangle inequality. The following zero-exponent observation requires no ambient ball-covering hypothesis.

**Proposition 5.1** (Subpolynomial growth). *Let  $M \subset K^{\times}$  be nonempty, with  $|x| \geq 1$  and every  $M(T)$  finite. If*

$$A_M(T) = T^{o(1)} \quad (T \rightarrow \infty),$$

*then  $\dim_B(M^{-1}) = 0$ .*

*Proof.* At scale  $\varepsilon$ , cover  $M(2/\varepsilon)^{-1}$  point by point. Every remaining reciprocal has absolute value less than  $\varepsilon/2$ , so the remaining set has diameter at most  $\varepsilon$ . Therefore

$$N(M^{-1}, \varepsilon) \leq A_M(2/\varepsilon) + 1 = \varepsilon^{-o(1)},$$

and the result follows.  $\square$

For  $\pi_1, \dots, \pi_k \in K^\times$ , write

$$\langle \pi_1, \dots, \pi_k \rangle := \{\pi_1^{m_1} \cdots \pi_k^{m_k} : m_1, \dots, m_k \in \mathbb{Z}_{\geq 0}\}.$$

**Corollary 5.2** (A finite-generation obstruction). *Let  $M \subset K^\times$  satisfy  $|x| \geq 1$ . Suppose that  $M^{-1}$  is totally bounded and  $\overline{\dim}_B(M^{-1}) > 0$ . Then there do not exist a finite set  $U \subset K^\times$  with  $|u| = 1$  for every  $u \in U$ , and elements  $\pi_1, \dots, \pi_k \in K^\times$  with  $|\pi_j| > 1$ , such that*

$$M \subset U \langle \pi_1, \dots, \pi_k \rangle.$$

*Proof.* Under such a containment,

$$A_M(T) \leq \#U \prod_{j=1}^k \left(1 + \frac{\log T}{\log |\pi_j|}\right) \ll (1 + \log T)^k.$$

Proposition 5.1 therefore gives  $\dim_B(M^{-1}) = 0$ , a contradiction.  $\square$

Equivalently, the containing reciprocal set is a finite union of products of contracting geometric rays.

No freeness or uniqueness of factorization is required in the preceding argument. Relations among generators can only decrease the number of distinct elements.

In particular, a monoid  $M \subset K^\times$  satisfying  $|x| \geq 1$  and having positive counting exponent cannot be finitely generated. If it were, its absolute-value-1 elements would form the finite set  $M(1)$ , while each remaining generator would have absolute value greater than 1. The preceding count would contradict positive counting exponent.

If there were only finitely many rational primes, then they would generate  $\mathbb{N}$ , contradicting (1). This recovers Saito's argument [Sai19]. The same reasoning applied to the monoid  $M_q$  in light of (15) proves that  $\mathbb{F}_q[t]$  has infinitely many monic irreducible polynomials. Applied to the nonzero Gaussian integers in light of (16), the argument proves that there are infinitely many Gaussian primes up to multiplication by units.

The applications just described require existence, but not uniqueness, of factorization into irreducibles; their infinitude conclusions also follow from the counting obstruction.

For a number field  $L$ , if the nonzero prime ideals of  $\mathcal{O}_L$  were  $\mathfrak{p}_1, \dots, \mathfrak{p}_k$ , then unique factorization of ideals would give

$$(S_L^{\text{pr}})^{-1} \subset S_L^{-1} \subset \prod_{j=1}^k \{\mathcal{N}(\mathfrak{p}_j)^{-m} : m \in \mathbb{Z}_{\geq 0}\};$$

the right-hand side has upper box dimension 0, contradicting Theorem 4.1.

*Remark 5.3* (Denominator zeta series). For  $M$  as in Theorem 1.1, define

$$\zeta_M(w) := \sum_{x \in M} |x|^{-w}, \quad |x|^{-w} := \exp(-w \log |x|) \quad (w \in \mathbb{C}).$$

Its abscissa of absolute convergence equals the counting exponent  $\alpha$ : dyadic summation gives convergence for  $\sigma = \Re w > \alpha$ , while for  $0 \leq \sigma < \alpha$  the sum over  $M(T)$  is at least  $T^{-\sigma} A_M(T) = T^{\alpha-\sigma+o(1)} \rightarrow \infty$ ; for  $\sigma < 0$ , the summands do not tend to 0. For uniformly discrete Euclidean denominator sets, this exponent agrees with Saito's upper zeta dimension [Sai18]. Proposition 3.3 shows that this abscissa, even together with uniform discreteness, does not determine reciprocal box dimension.

**5.2. A closing question.** Our sparse examples show that counting growth and uniform discreteness need not force critical-scale packing. Thus, we may ask the following.

**Question.** Which verifiable algebraic or arithmetic hypotheses imply the annular estimate (11), or at least all-scale critical-scale packing, for families without an explicit lattice model?

Any such criterion must control metric concentration at the height-dependent critical scale, not merely exclude exact algebraic containment.

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