

CONSECUTIVE DIVISORS OF SUMS OF FIBONACCI POLYNOMIALS: A LUCA–MARQUES ANALOGUE IN FINITE-FIELD POLYNOMIAL RINGS

SCOTT DUKE KOMINERS

ABSTRACT. We construct an infinite family of pairs of consecutive monic polynomials in $\mathbb{F}_q[T]$ such that both polynomials divide the same Fibonacci-polynomial sum—specifically, the sum whose upper index is one less than the pair’s common norm. This gives a function-field counterpart of the divisibility condition in the Luca–Marques theorem on consecutive integral arithmetic means of Fibonacci numbers.

1. INTRODUCTION

Answering a question of Fatehizadeh and Yaqub [YF22], Luca and Marques [LM25] proved that there are infinitely many positive integers n for which both

$$\frac{1}{n} \sum_{j=1}^n F_j \quad \text{and} \quad \frac{1}{n+1} \sum_{j=1}^{n+1} F_j$$

are integers, where F_n denotes the Fibonacci sequence, with $F_0 = 0$ and $F_1 = 1$.¹ Since

$$\sum_{j=1}^n F_j = F_{n+2} - 1,$$

the Luca–Marques [LM25] result is equivalently the simultaneous congruence statement

$$n \mid F_{n+2} - 1, \quad n+1 \mid F_{n+3} - 1$$

for infinitely many n .

This note proves a norm-indexed polynomial analogue over finite fields. Fix a prime power q with $\gcd(q, 6) = 1$. In $\mathbb{F}_q[T]$, define the Fibonacci-polynomial sequence

$$G_0 = 0, \quad G_1 = 1, \quad G_n = TG_{n-1} + G_{n-2} \quad (n \geq 2),$$

2020 *Mathematics Subject Classification*. Primary 11B39; Secondary 11T55, 11B83.

Key words and phrases. Fibonacci polynomials, Lucas polynomials, finite fields, function fields, divisibility, polynomial recurrences.

I used LLMs to assist with computations and analysis in the preparation of this article, particularly GPT-5.5 Pro and Claude 4.7 Opus (accessed in part via Poe with the support of Quora, where I am an advisor). The problem, methods, and final written form are my own, and of course any errors remain my responsibility. This work was conducted while I was visiting the Technological Innovation, Entrepreneurship, and Strategic Management (TIES) Group at the MIT Sloan School of Management; I greatly appreciate their hospitality.

¹See also [FC25] for related results.

and its Lucas companion

$$L_0 = 2, \quad L_1 = T, \quad L_n = TL_{n-1} + L_{n-2} \quad (n \geq 2).$$

For a nonzero polynomial $A \in \mathbb{F}_q[T]$, put

$$|A| = q^{\deg A}.$$

For $N \geq 0$, set

$$\mathcal{E}_N(T) := G_{N+1}(T) + G_N(T) - 1.$$

The recurrence defining G_j gives

$$TG_j = G_{j+1} - G_{j-1} \quad (j \geq 1),$$

and hence, with the empty-sum convention,

$$\mathcal{E}_N(T) = T \sum_{j=1}^N G_j(T) \quad (N \geq 0).$$

We say that a nonzero polynomial $A \in \mathbb{F}_q[T]$ is *norm-Fibonacci-summable* if

- (1) $\gcd(A, T) = 1$, and
- (2) $A \mid \mathcal{E}_{|A|-1}(T)$.

Since $\mathcal{E}_N = T \sum_{j=1}^N G_j(T)$, the coprimality condition $\gcd(A, T) = 1$ makes the second condition equivalent to

$$A \mid \sum_{j=1}^{|A|-1} G_j(T).$$

Our main result is the following explicit construction.

Theorem 1.1. *Let q be a prime power with $\gcd(q, 6) = 1$. For every integer $a \geq 1$, set $k = 2^a$ and*

$$B = L_{2k}.$$

Then B is monic, $B + 1$ is monic, and both are norm-Fibonacci-summable over $\mathbb{F}_q[T]$. More precisely,

$$B + 1 = \frac{G_{3k}}{G_k},$$

and, with

$$N = q^{2k} - 1 = q^{\deg B} - 1 = q^{\deg(B+1)} - 1,$$

we have

$$B(B + 1) \mid \sum_{j=1}^N G_j(T).$$

In particular, there are infinitely many monic polynomials $B \in \mathbb{F}_q[T]$ such that both B and $B + 1$ are norm-Fibonacci-summable.

The proof separates into two elementary parts. The polynomial part is a period-doubling lemma: if a polynomial divides G_r with r even, then it divides $\mathcal{E}_N$ whenever $2r \mid N$. The integer part is the simple observation that, for $k = 2^a$, the hypothesis $\gcd(q, 6) = 1$ implies

$$24k \mid q^{2k} - 1.$$

Together the preceding facts force the required divisibilities for

$$B = L_{2k} \quad \text{and} \quad B + 1 = \frac{G_{3k}}{G_k}.$$

2. PRELIMINARIES

All polynomial divisibilities and congruences in what follows are taken in $\mathbb{F}_q[T]$. Integer divisibilities, such as $24k \mid q^{2k} - 1$, are taken in $\mathbb{Z}$. Congruences between matrices are understood entrywise.

A direct induction shows that G_n is monic of degree $n - 1$ for $n \geq 1$, and that L_n is monic of degree n for $n \geq 1$.

Set

$$M = \begin{pmatrix} T & 1 \\ 1 & 0 \end{pmatrix} \in \text{GL}_2(\mathbb{F}_q[T]).$$

Then

$$M^n = \begin{pmatrix} G_{n+1} & G_n \\ G_n & G_{n-1} \end{pmatrix} \quad (n \geq 1).$$

Taking determinants gives Cassini's identity

$$G_{n+1}G_{n-1} - G_n^2 = (-1)^n.$$

In a splitting field of $X^2 - TX - 1$ over $\mathbb{F}_q(T)$, let α and β be the two roots. Then

$$\alpha + \beta = T, \quad \alpha\beta = -1, \quad \beta = -\alpha^{-1}.$$

Because $\gcd(q, 6) = 1$, the characteristic of $\mathbb{F}_q$ is not 2, so the roots are distinct. The standard Binet formulas give

$$G_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad L_n = \alpha^n + \beta^n.$$

Thus

$$G_{2m} = G_m L_m.$$

Also, for $k \geq 1$, the identity $x^3 - y^3 = (x - y)(x^2 + xy + y^2)$ gives

$$G_{3k} = G_k \left(\alpha^{2k} + (\alpha\beta)^k + \beta^{2k} \right).$$

Hence $G_k \mid G_{3k}$, and

$$\frac{G_{3k}}{G_k} = \alpha^{2k} + (\alpha\beta)^k + \beta^{2k} = L_{2k} + (-1)^k.$$

For even k , this becomes

$$\frac{G_{3k}}{G_k} = L_{2k} + 1.$$

Henceforth, for even k , set

$$B_k := L_{2k}, \quad C_k := B_k + 1 = \frac{G_{3k}}{G_k}.$$

Both B_k and C_k are monic of degree $2k$. Also, evaluating the Lucas recurrence at $T = 0$ gives

$$B_k(0) = L_{2k}(0) = 2, \quad C_k(0) = 3.$$

Because $\gcd(q, 6) = 1$, both constants are nonzero in $\mathbb{F}_q$. Therefore

$$\gcd(B_k, T) = \gcd(C_k, T) = 1.$$

Moreover,

$$|B_k| - 1 = |C_k| - 1 = q^{2k} - 1.$$

3. A PERIOD-DOUBLING LEMMA

Lemma 3.1. *Let $H \in \mathbb{F}_q[T]$ be nonzero, and let r be a positive even integer. If $H \mid G_r$, then*

$$M^{2r} \equiv I \pmod{H}.$$

Consequently, for every positive integer N with $2r \mid N$, we have

$$H \mid \mathcal{E}_N.$$

Proof. Since $H \mid G_r$, the matrix formula for M^r gives

$$M^r \equiv \begin{pmatrix} G_{r+1} & 0 \\ 0 & G_{r-1} \end{pmatrix} \pmod{H}.$$

The recurrence gives

$$G_{r+1} - G_{r-1} = TG_r,$$

so that

$$G_{r+1} \equiv G_{r-1} \pmod{H}.$$

Hence, we have

$$M^r \equiv G_{r+1}I \pmod{H}.$$

Taking determinants yields

$$G_{r+1}^2 \equiv \det(M^r) = (-1)^r = 1 \pmod{H},$$

because r is even. Therefore

$$M^{2r} \equiv I \pmod{H}.$$

If $2r \mid N$, then $M^N \equiv I \pmod{H}$. Comparing entries in

$$M^N = \begin{pmatrix} G_{N+1} & G_N \\ G_N & G_{N-1} \end{pmatrix}$$

gives

$$G_N \equiv 0 \pmod{H}, \quad G_{N+1} \equiv 1 \pmod{H}.$$

Thus, we have

$$\mathcal{E}_N = G_{N+1} + G_N - 1 \equiv 0 \pmod{H},$$

as required. □

Remark. The factor of 2 in Lemma 3.1 is essential. The congruence $H \mid G_r$ forces only

$$G_{r+1}^2 \equiv 1 \pmod{H},$$

not necessarily

$$G_{r+1} \equiv 1 \pmod{H}.$$

4. AN ELEMENTARY INTEGER INPUT

Lemma 4.1. *Let q be a prime power with $\gcd(q, 6) = 1$. If $a \geq 1$ and $k = 2^a$, then*

$$24k \mid q^{2k} - 1.$$

Proof. Since q is odd,

$$q^2 \equiv 1 \pmod{8}.$$

We use the elementary observation that if $x \equiv 1 \pmod{2^m}$ with $m \geq 3$, then

$$x^2 \equiv 1 \pmod{2^{m+1}}.$$

Indeed, writing $x = 1 + 2^m u$, we have

$$x^2 - 1 = 2^{m+1}u + 2^{2m}u^2,$$

which is divisible by 2^{m+1} .

Starting from $q^2 \equiv 1 \pmod{8}$ and iterating this observation a times gives

$$(q^2)^{2^a} = q^{2k} \equiv 1 \pmod{2^{a+3}}.$$

Since $3 \nmid q$, we also have

$$q^2 \equiv 1 \pmod{3},$$

and hence

$$q^{2k} \equiv 1 \pmod{3}.$$

Because 2^{a+3} and 3 are coprime, it follows that

$$3 \cdot 2^{a+3} \mid q^{2k} - 1.$$

But of course

$$3 \cdot 2^{a+3} = 24 \cdot 2^a = 24k,$$

so we have

$$24k \mid q^{2k} - 1. \quad \square$$

5. A SUFFICIENT CRITERION

Proposition 5.1. *Let q be a prime power with $\gcd(q, 6) = 1$, and let k be an even positive integer such that*

$$24k \mid q^{2k} - 1,$$

and set

$$B_k = L_{2k}, \quad C_k = B_k + 1 = \frac{G_{3k}}{G_k}.$$

Then B_k and C_k are norm-Fibonacci-summable. More precisely, with

$$N = q^{2k} - 1,$$

we have

$$B_k C_k \mid \sum_{j=1}^N G_j(T).$$

Proof. As noted in Section 2, both B_k and C_k are monic of degree $2k$; both are coprime to T ; and

$$|B_k| - 1 = |C_k| - 1 = q^{2k} - 1 = N.$$

First consider

$$C_k = \frac{G_{3k}}{G_k} = B_k + 1.$$

Since $C_k \mid G_{3k}$, and since $3k$ is even, Lemma 3.1 applies with

$$H = C_k, \quad r = 3k.$$

Because

$$2r = 6k \mid 24k \quad \text{and} \quad 24k \mid N,$$

we obtain

$$C_k \mid \mathcal{E}_N.$$

Next, the identity

$$G_{4k} = G_{2k} L_{2k}$$

shows that

$$B_k = L_{2k} \mid G_{4k}.$$

Since $4k$ is even, Lemma 3.1 applies with

$$H = B_k, \quad r = 4k.$$

Because

$$2r = 8k \mid 24k \quad \text{and} \quad 24k \mid N,$$

we obtain

$$B_k \mid \mathcal{E}_N.$$

Now, since

$$\mathcal{E}_N = T \sum_{j=1}^N G_j(T),$$

and B_k and C_k are each coprime to T , the divisibilities

$$B_k \mid \mathcal{E}_N \quad \text{and} \quad C_k \mid \mathcal{E}_N$$

imply that

$$B_k \mid \sum_{j=1}^N G_j(T) \quad \text{and} \quad C_k \mid \sum_{j=1}^N G_j(T).$$

Finally, $C_k = B_k + 1$, so $\gcd(B_k, C_k) = 1$. Therefore

$$B_k C_k \mid \sum_{j=1}^N G_j(T).$$

In particular, B_k and C_k are norm-Fibonacci-summable. □

6. PROOF OF THEOREM 1.1

Let $a \geq 1$, and set $k = 2^a$. Then k is even, and Lemma 4.1 gives

$$24k \mid q^{2k} - 1.$$

Proposition 5.1 therefore applies to

$$B = L_{2k};$$

this proves that B and

$$B + 1 = \frac{G_{3k}}{G_k}$$

are both norm-Fibonacci-summable, and indeed that

$$B(B + 1) \mid \sum_{j=1}^{q^{2k}-1} G_j(T).$$

Finally,

$$\deg B = \deg L_{2k} = 2k = 2^{a+1},$$

which tends to infinity as $a \rightarrow \infty$. Hence the construction gives infinitely many distinct monic polynomials $B \in \mathbb{F}_q[T]$ with the desired property. $\square$

7. AN EXAMPLE: $q = 5$, $k = 2$

Let $q = 5$ and $a = 1$, so $k = 2$. Then

$$N = 5^4 - 1 = 624, \quad 24k = 48 \mid 624.$$

The construction gives

$$B_2 = L_4 = T^4 + 4T^2 + 2, \quad B_2 + 1 = T^4 + 4T^2 + 3$$

in $\mathbb{F}_5[T]$. Moreover,

$$B_2 + 1 = \frac{G_6}{G_2}.$$

Since $B_2 \mid G_8$ and $B_2 + 1 \mid G_6$ —and $16 \mid 624$ and $12 \mid 624$ —Lemma 3.1 gives

$$B_2 \mid \mathcal{E}_{624}, \quad B_2 + 1 \mid \mathcal{E}_{624};$$

their product is

$$B_2(B_2 + 1) = T^8 + 3T^6 + T^4 + 1.$$

As both factors are coprime to T , it follows that

$$T^8 + 3T^6 + T^4 + 1 \mid \sum_{j=1}^{624} G_j(T)$$

in $\mathbb{F}_5[T]$.

8. REMARKS

Our construction for finite-field polynomial rings is simpler than its integer counterpart in one important respect: the consecutive pair is produced by an identity before any congruence argument is made. For even k ,

$$\frac{G_{3k}}{G_k} = L_{2k} + 1,$$

so choosing

$$B_k = L_{2k}$$

automatically gives the consecutive pair

$$B_k, B_k + 1.$$

The remaining work is only to force both members of the pair to divide the same norm-indexed sum.

The period-doubling lemma reduces the required polynomial congruences to two integer-divisibility conditions. Indeed,

$$B_k + 1 = \frac{G_{3k}}{G_k} \mid G_{3k},$$

so $B_k + 1 \mid \mathcal{E}_N$ follows once

$$6k \mid N.$$

Similarly,

$$B_k = L_{2k} \mid G_{4k},$$

so $B_k \mid \mathcal{E}_N$ follows once

$$8k \mid N.$$

Thus it is enough to have

$$\text{lcm}(6k, 8k) = 24k \mid N.$$

For $k = 2^a$, Lemma 4.1 supplies exactly this with

$$N = q^{2k} - 1.$$

No primitive-divisor theorem is needed for this explicit infinite family.

The sufficient criterion in Proposition 5.1 is broader than the power-of-two family used in Theorem 1.1: any even positive integer k satisfying

$$24k \mid q^{2k} - 1$$

gives a version of the same construction. The choice $k = 2^a$ is used because it gives a uniform elementary infinite family for every q with $\gcd(q, 6) = 1$.

The hypothesis $\gcd(q, 6) = 1$ enters the argument in several elementary ways. The condition $\text{char}(\mathbb{F}_q) \neq 2$ ensures that the roots of $X^2 - TX - 1$ are distinct and also gives

$$q^2 \equiv 1 \pmod{8}.$$

The condition $\text{char}(\mathbb{F}_q) \neq 3$ gives

$$q^2 \equiv 1 \pmod{3}.$$

Together these congruences supply the factor $24k$ in Lemma 4.1. The same hypothesis also ensures that

$$B_k(0) = 2 \quad \text{and} \quad (B_k + 1)(0) = 3$$

are nonzero in $\mathbb{F}_q$, so both B_k and $B_k + 1$ are coprime to T . In characteristics 2 or 3, both the coprimality condition and the period-forcing argument would require modification.

Finally, we note that the choice of the index $|A| - 1$, rather than $|A|$, is not merely an artifact of the construction. Appendix A shows that, for q odd and $d \geq 1$, the polynomial $\sum_{j=1}^{q^d} G_j$ has no divisor of degree d ; equivalently, no polynomial A of degree d and coprime to T can satisfy

$$A \mid \mathcal{E}_{|A|}.$$

REFERENCES

- [FC25] Oisín Flynn-Connolly, *On the divisibility of sums of Fibonacci numbers*, INTEGERS **25** (2025), Article A87.
- [LM25] Florian Luca and Diego Marques, *On integral consecutive arithmetic means of the first Fibonacci numbers*, Archiv der Mathematik **125** (2025), no. 5, 505–511.
- [YF22] Daniel Yaquobi and Amirali Fatehizadeh, *Average of the Fibonacci numbers*, Journal of Integer Sequences **25** (2022), Article 22.2.6.

APPENDIX A. A SHIFTED-INDEX OBSTRUCTION

For $N \geq 1$, write

$$S_N(T) = \sum_{j=1}^N G_j(T).$$

Proposition A.1. *Assume that q is odd. Let $d \geq 1$, and put $Q = q^d$. Then S_Q has no divisor of degree d in $\mathbb{F}_q[T]$. Consequently, no polynomial $A \in \mathbb{F}_q[T]$ with $\deg A = d$ and $\gcd(A, T) = 1$ satisfies*

$$A \mid \mathcal{E}_{|A|}.$$

Proof. The key point is that every root of S_Q lies in $\mathbb{F}_{q^{2d}}$, but no root lies in $\mathbb{F}_{q^d}$. We first prove this root-field statement, and then translate it into a degree obstruction.

Put

$$D = T^2 + 4, \quad h = \frac{Q - 1}{2}.$$

Let α and β be the roots of $X^2 - TX - 1$ in a splitting field over $\mathbb{F}_q(T)$. Since Q is a power of $\text{char}(\mathbb{F}_q)$, the Q -power map is Frobenius. Hence the Binet formulas give

$$G_Q = \frac{\alpha^Q - \beta^Q}{\alpha - \beta} = \frac{(\alpha - \beta)^Q}{\alpha - \beta} = (\alpha - \beta)^{Q-1} = D^h,$$

and

$$L_Q = \alpha^Q + \beta^Q = (\alpha + \beta)^Q = T^Q.$$

The identity

$$2G_{n+1} = L_n + TG_n$$

therefore gives

$$G_{Q+1} = \frac{T^Q + TD^h}{2}.$$

Also, summing $TG_j = G_{j+1} - G_{j-1}$ from $j = 1$ to Q gives

$$TS_Q = G_{Q+1} + G_Q - 1.$$

Combining these identities, we obtain

$$(1) \quad TS_Q = \frac{T^Q + (T+2)D^h - 2}{2}.$$

At $T = 0$, the recurrence gives $G_{2m}(0) = 0$ and $G_{2m+1}(0) = (-1)^m$. Since Q is odd, direct summation gives

$$S_Q(0) = \sum_{m=0}^{(Q-1)/2} (-1)^m = \frac{1}{2}$$

in $\mathbb{F}_q$. Thus $T \nmid S_Q$.

Now let P be an irreducible factor of S_Q , let

$$r = \deg P,$$

and let $\theta \in \overline{\mathbb{F}_q}$ be a root of P . Since $T \nmid S_Q$, we have $\theta \neq 0$. Set

$$\delta = \theta^2 + 4, \quad u = \delta^h.$$

Evaluating (1) at $T = \theta$ gives

$$(2) \quad \theta^Q + (\theta + 2)u - 2 = 0.$$

We first rule out $\delta = 0$. If $\delta = 0$, then $u = 0$, and (2) gives $\theta^Q = 2$. Squaring and using $\theta^2 = -4$, we obtain

$$4 = \theta^{2Q} = (-4)^Q = -4,$$

which is impossible because q is odd. Therefore

$$\delta \neq 0.$$

Since $\delta \neq 0$, we have

$$u^2 = \delta^{Q-1} = \frac{\delta^Q}{\delta} = \frac{\theta^{2Q} + 4}{\theta^2 + 4}.$$

Using $\theta^Q = 2 - (\theta + 2)u$ from (2), this becomes

$$(\theta^2 + 4)u^2 = (2 - (\theta + 2)u)^2 + 4.$$

After simplification,

$$0 = 4(\theta u^2 - (\theta + 2)u + 2) = 4(u - 1)(\theta u - 2).$$

Since q is odd, either

$$u = 1 \quad \text{or} \quad \theta u = 2.$$

The root equation (2) therefore yields

$$\theta^Q = \begin{cases} -\theta, & \text{if } u = 1, \\ -4/\theta, & \text{if } \theta u = 2. \end{cases}$$

In either case, applying the Q -power map once more gives

$$\theta^{Q^2} = \theta.$$

Indeed, if $\theta^Q = -\theta$, then

$$\theta^{Q^2} = (-\theta)^Q = -\theta^Q = \theta,$$

while if $\theta^Q = -4/\theta$, then

$$\theta^{Q^2} = \left(-\frac{4}{\theta}\right)^Q = -\frac{4}{\theta^Q} = -\frac{4}{-4/\theta} = \theta.$$

Thus

$$\theta \in \mathbb{F}_{q^{2d}}.$$

On the other hand, $\theta \notin \mathbb{F}_{q^d}$. For if $\theta^Q = \theta$, then the alternative $u = 1$ would give

$$\theta = -\theta,$$

hence $\theta = 0$, contradicting $\theta \neq 0$. The alternative $\theta u = 2$ would give

$$\theta = -\frac{4}{\theta},$$

hence $\theta^2 = -4$, contradicting $\delta \neq 0$. Therefore

$$\theta \in \mathbb{F}_{q^{2d}} \setminus \mathbb{F}_{q^d}.$$

Since θ has degree r over $\mathbb{F}_q$, the standard finite-field criterion gives

$$r \mid 2d \quad \text{but} \quad r \nmid d.$$

Write

$$d = 2^\nu d_0, \quad d_0 \text{ odd}.$$

If $r \mid 2d$ but $r \nmid d$, then $2^{\nu+1} \mid r$. Hence every irreducible factor of S_Q has degree divisible by $2^{\nu+1}$. It follows that the degree of every nonzero divisor of S_Q is divisible by $2^{\nu+1}$. But $d = 2^\nu d_0$ is not divisible by $2^{\nu+1}$. Therefore S_Q has no divisor of degree d .

For the final assertion, let $A \in \mathbb{F}_q[T]$ have degree d and satisfy $\gcd(A, T) = 1$. Then

$$|A| = q^d = Q$$

and

$$\mathcal{E}_{|A|} = \mathcal{E}_Q = TS_Q.$$

If $A \mid \mathcal{E}_Q$, then $A \mid TS_Q$. Since $\gcd(A, T) = 1$, this would imply $A \mid S_Q$, contradicting the first part of the proposition. $\square$

HARVARD BUSINESS SCHOOL; DEPARTMENT OF ECONOMICS AND CENTER OF MATHEMATICAL SCIENCES AND APPLICATIONS, HARVARD UNIVERSITY; AND A16Z CRYPTO

Email address: `kominers@fas.harvard.edu`