Timid Signed Quasifibonacci Products

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Abstract

We show that a finite or infinite product $\prod_k(1+\varepsilon_k x^{u_k})$ with quasifibonacci exponents of level $L \geq 2$ has coefficients in $\{-1,0,1\}$ if and only if every block of $L+1$ consecutive signs has product -1. When this rule first fails, we locate the lowest-degree coefficient outside this set; it equals ± 2 and remains unchanged in every subsequent partial product.

1 Introduction

Following Zhao [18], we call a polynomial or formal power series *timid* if each of its coefficients belongs to $\{-1,0,1\}$. A classical example is the product

$$A_F(x) := \prod_{k\geq 1}(1-x^{F_k}) = (1-x)(1-x^2)(1-x^3)(1-x^5)\cdots = 1-x-x^2+x^4+x^7-x^8+\cdots$$

with Fibonacci exponents ($F_1 = 1$, $F_2 = 2$, and $F_{k+2} = F_{k+1} + F_k$): $A_F(x)$ is timid, as are all its finite partial products [16].

We ask which choices of signs preserve the timidity property. The first equality between two distinct subset sums, $1 + 2 = 3$, already suggests a necessary cancellation rule. We have

$$[x^3](1+\varepsilon_1 x)(1+\varepsilon_2 x^2)(1+\varepsilon_3 x^3) = \varepsilon_1\varepsilon_2 + \varepsilon_3;$$

because both summands are signs, their sum belongs to $\{-1,0,1\}$ exactly when they cancel, or equivalently when

$$\varepsilon_1\varepsilon_2\varepsilon_3 = -1.$$

Surprisingly, we find that imposing the same rule on every consecutive sign triple precisely characterizes timidity: For Fibonacci exponents, a finite product is timid if and only if its signs satisfy the rule $\varepsilon_k\varepsilon_{k+1}\varepsilon_{k+2} = -1$, which means that the only infinite patterns that work are

$$(-,-,-,\ldots), \quad (-,+,+,-,+,+,\ldots), \quad (+,-,+,+,-,+,\ldots), \quad \text{and} \quad (+,+,-,+,+,-,\ldots).$$

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More generally, an analogous rule gives a complete classification when each exponent is the sum of the preceding L exponents.

We also determine exactly how the classification fails: at the first violation of the sign rule, we locate the lowest-degree coefficient outside $\{-1, 0, 1\}$, compute its value, and show that no choice of later signs can change it.

1.1 Setting and main results

Fix an integer $L \geq 2$. In the formulation used by Diao [6, Definition 1.1], a *quasifibonacci sequence of level* L is a sequence of positive integers $(u_k)_{k\geq 1}$ satisfying

$$\begin{aligned} u_j &> \sum_{i=1}^{j-1} u_i \qquad (1 \leq j \leq L), \\ u_{k+L} &= \sum_{i=k}^{k+L-1} u_i \quad (k \geq 1). \end{aligned} \tag{1}$$

(An empty sum is taken to be zero.) The first line of (1) says that the initial block is *superincreasing*: each term exceeds the sum of all earlier terms in that block. The second line of (1) is the classical L-step Fibonacci recurrence; see Miles [10] and Wolfram [17].

By construction, (u_k) is strictly increasing and tends to infinity. At level 2, the initial inequalities are simply $0 < u_1 < u_2$. No coprimality assumption is imposed.

For signs $\varepsilon_k \in \{-1, 1\}$, we define

$$P_n^{\varepsilon}(x) = \prod_{k=1}^{n}(1 + \varepsilon_k x^{u_k}), \qquad P_0^{\varepsilon}(x) = 1, \qquad P_\infty^{\varepsilon}(x) = \prod_{k\geq 1}(1 + \varepsilon_k x^{u_k}).$$

The infinite product P_∞^{ε} is well defined in $\mathbb{Z}[[x]]$, because only finitely many factors can affect the coefficient of any fixed power of x. We write

$$f[m] = [x^m]f, \qquad \operatorname{supp}(f) = \{m \geq 0 : f[m] \neq 0\}.$$

We set $f[m] = 0$ for $m < 0$, and also for $m > \deg f$ when f is a polynomial. When the signs are fixed, we suppress the superscript on P_n^{ε}, writing simply P_n.

Each coefficient is a signed count of distinct-part representations:

$$P_n^{\varepsilon}[m] = \sum_{\substack{S\subseteq\{1,\dots,n\} \\ \sum_{i\in S} u_i = m}} \prod_{i\in S} \varepsilon_i. \tag{2}$$

Thus a representation is a subset of the available indices, and its weight is the product of the signs at those indices. In particular, for the all-minus sign choice, the coefficient counts representations with an even number of parts minus representations with an odd number of parts.

The recurrence gives the equality

$$u_k + \cdots + u_{k+L-1} = u_{k+L}.$$

Whenever a representation contains all the parts on the left and does not contain the part on the right, we may replace those L parts by u_{k+L}. The weights of the two representations are opposite exactly when

$$\prod_{i=k}^{k+L-1} \varepsilon_i = -\varepsilon_{k+L},$$

or equivalently when

$$\prod_{i=k}^{k+L} \varepsilon_i = -1. \tag{3}$$

We call an infinite sign sequence *admissible* if (3) holds for every $k \geq 1$; a finite sign word is admissible if the condition holds for every block of $L+1$ consecutive signs contained in the word. In particular, every word of length at most L is admissible.

Our first main result is as follows.

Theorem 1 (Classification)**.** *Let (u_k) satisfy (1). For every $N \geq L$, the following are equivalent:*

(i) *P_N^ε is timid;*

(ii) *P_n^ε is timid for every $1 \leq n \leq N$;*

(iii) *$(\varepsilon_1, \ldots, \varepsilon_N)$ is admissible.*

For each $N \geq L$, exactly 2^L words of length N satisfy these conditions. Each extends uniquely to an admissible infinite sequence.

The admissible infinite sequences are precisely those obtained by repeating a block of $L+1$ signs with product -1. There are 2^L such sequences, and each has period dividing $L+1$. Moreover, P_∞^ε is timid if and only if ε is admissible, equivalently if and only if every partial product is timid.

For $1 \leq N < L$, every one of the 2^N sign words gives a timid product.

Equivalently, for timidity, the repeating block must contain an odd number of minus signs. The all-minus sequence is therefore admissible if and only if L is even. At every level, a block with one minus sign and L plus signs gives an admissible sequence.

The classification identifies all sign choices that yield timid products. Our second main result locates the first non-timid coefficient for every other sign choice.

Theorem 2 (First failure and persistence)**.** *Suppose $r \geq L+1$, the word $(\varepsilon_1, \ldots, \varepsilon_{r-1})$ is admissible, and*

$$\prod_{i=r-L}^{r} \varepsilon_i = 1.$$

Thus r is the index at which the sign rule first fails.

Define d_r by the following table, where $1 \leq j \leq L$:

$r \bmod 2L$	d_r	$P_N[u_r + d_r]$
j	u_j	$2\varepsilon_r\varepsilon_j$
$L+j$	0	$2\varepsilon_r$

In the second row, $L + j = 2L$ denotes residue 0.

For every continuation of the first r signs and every finite $N \geq r$, the least degree m with $|P_N[m]| > 1$ is

$$m = u_r + d_r,$$

and its coefficient is the value displayed in the table. The same assertion holds for an infinite continuation and $N = \infty$.

Moreover,

$$u_r + d_r < u_{r+1},$$

and the maximum absolute value of a coefficient of P_r is exactly 2.

For example, with Fibonacci exponents $1, 2, 3, 5, 8, \ldots$, the sign word

$$(-, -, -, -, +)$$

first violates the rule at $r = 5$. The theorem gives $d_5 = 1$: every finite or infinite continuation has its least non-timid coefficient at degree $u_5 + 1 = 9$, where the coefficient is -2. Since the next exponent is 13, later factors cannot alter this coefficient or any coefficient below it.

1.2 Relation to prior work

There are several known proofs of timidity for the all-minus Fibonacci product. White [16] proved both the finite- and infinite-product assertions using a Lucas-number expansion and coefficient symmetry. Robbins [11] and Ardila [1] later proved the infinite assertion using, respectively, recurrences based on Zeckendorf representations and a three-block coefficient recursion derived from partition bijections. Weinstein [15] also obtained timidity from a Zeckendorf-based formula for the generating polynomial that records the number of parts. Zhao [18, Proposition 2] gave an especially short proof of the finite-product assertion for every sequence with positive initial terms $u_1 < u_2$ satisfying $u_{k+2} = u_{k+1} + u_k$, by way of an elegant simultaneous induction with two auxiliary polynomials.

For higher levels, Diao [6, Theorem 1.1] proved timidity of the all-minus infinite product whenever L is even. His proof uses recursive decompositions of partially ordered sets of quasifibonacci representations. The corresponding finite assertion follows from coefficient reciprocity and a degree bound; we give that deduction in Section 2.

At odd levels, the all-minus conclusion itself is false—already at the first recurrence step, we have

$$[x^{u_{L+1}}] \prod_{k=1}^{L+1} (1 - x^{u_k}) = (-1)^L - 1 = -2.$$

Diao [6, Section 5.3] observed the stronger fact that the coefficients of the all-minus infinite product are unbounded in absolute value at odd levels. Our result does not extend the all-minus assertion to those levels. Instead, it identifies exactly which position-dependent choices of signs give timid products, at both even and odd levels, and locates a persistent coefficient of magnitude 2 whenever that sign condition first fails.

There is also literature describing the individual coefficients of the all-minus Fibonacci product,

$$a_F(m) := [x^m] A_F(x) = [x^m] \prod_{k \geq 1} (1 - x^{F_k}).$$

Ardila [1] gave a recursive description of $a_F(m)$. Shallit [13] obtained an automaton-based characterization using Berstel's transducer [2]. Eğecioğlu [7] gave coefficient formulas and matrix descriptions, using an auxiliary bivariate product; his commentary [8] records earlier timidity results. Related work on unsigned representation counts and weighted products is discussed in Section 4.2.

1.3 Proof idea

Our proof of Theorem 1 works with coefficients, which aggregate the weights of all representations of a degree, rather than constructing a pairing of individual representations.

Why the sign rule is sufficient. Adding the n-th factor gives

$$P_n[m] = P_{n-1}[m] + \varepsilon_n P_{n-1}[m - u_n]. \tag{4}$$

We say that there is a *coefficient overlap at degree m* in this recursion if

$$P_{n-1}[m] \neq 0 \quad \text{and} \quad P_{n-1}[m - u_n] \neq 0.$$

If P_{n-1} is timid, only such a degree can produce a coefficient outside $\{-1, 0, 1\}$: two nonzero contributions must cancel rather than add to ± 2.

Suppose $n > L$ and put $q = n - L$. The growth conditions imply that every representation of a degree at least u_n using the first $n - 1$ exponents must include

$$u_{q+1}, \ldots, u_{n-1}.$$

Removing these required parts reduces the two coefficients in an overlap to coefficients of the earlier product P_q, at degrees $u_q + t$ and t. A short coefficient-pair lemma determines their relative signs from the factor $1 + \varepsilon_q x^{u_q}$. The admissibility condition then says exactly that the two contributions in (4) have opposite signs. This proves sufficiency by induction.

Locating the first failure and proving persistence. Necessity in Theorem 1 follows from the stronger first-failure statement in Theorem 2. At the first violating index r, the signs agree through $r - 1$ with a unique admissible sequence, but the r-th sign is reversed. Every coefficient overlap at that step therefore changes from a cancellation into a coefficient of magnitude 2.

The remaining issue is to find the earliest such overlap and show that later factors cannot remove it. For this we examine coefficients at degrees $u_n + t$, where t ranges over the proper subset sums of the first L exponents. For an admissible product, the sets of offsets with nonzero coefficients are complemented each time n increases by L; they therefore repeat after $2L$ indices. This finite support calculation identifies the first coefficient of magnitude 2 at degree $u_r + d_r$, where

$$d_r \in \{0, u_1, \ldots, u_L\}$$

is given explicitly in Theorem 2. We also show that $u_r + d_r < u_{r+1}$. Every later factor has exponent at least u_{r+1}, so neither that coefficient nor any lower coefficient can change.

2 Cancellation and sufficiency

We first record the elementary structure of admissible signs.

Lemma 3 (Admissible sign sequences)**.** *The admissible infinite sequences are precisely those obtained by periodically repeating a block of $L + 1$ signs with product -1. Every admissible word of length at least L extends uniquely to an admissible infinite sequence. There are 2^L such sequences and 2^L admissible words of each length $N \geq L$.*

Proof. The sign rule (3) is equivalent to

$$\varepsilon_{k+L} = -\prod_{i=k}^{k+L-1} \varepsilon_i \qquad (k \geq 1). \tag{5}$$

Thus the first L signs determine a unique admissible sequence. This sequence agrees with any admissible word having those initial signs.

Comparing consecutive instances of (3) gives

$$\varepsilon_{k+L+1} = \varepsilon_k.$$

Conversely, in a sequence obtained by repeating a block of $L+1$ signs, every consecutive block of that length is a cyclic shift of the initial block. If the initial block has product -1, the sequence is admissible. The 2^L choices of the first L signs give both counts. □

Put

$$D_0 = 0, \qquad D_n = \sum_{i=1}^{n} u_i = \deg P_n \quad (n \geq 1).$$

We repeatedly use the following bounds.

Lemma 4 (Degree bounds)**.** *For the exponents in* (1)*,*

$$D_{j-2} < u_j \quad (j \geq 2), \qquad D_{j-1} < 2u_j \quad (j \geq 1). \tag{6}$$

Proof. For $2 \leq j \leq L$, the first bound follows from the initial inequalities. For $j = L+1$, we have

$$u_{L+1} = D_L > D_{L-1}.$$

For $j \geq L+2$, induction gives

$$u_j \geq u_{j-1} + u_{j-2} > D_{j-3} + u_{j-2} = D_{j-2}.$$

This proves the first bound. For $j \geq 2$, adding $u_{j-1} < u_j$ gives

$$D_{j-1} = D_{j-2} + u_{j-1} < 2u_j.$$

The case $j = 1$ is immediate from $D_0 = 0$. □

Reciprocity and finite products. For any signs, taking complementary subsets in (2) gives

$$P_N^{\varepsilon}[m] = \left(\prod_{i=1}^{N} \varepsilon_i\right) P_N^{\varepsilon}[D_N - m] \qquad (0 \leq m \leq D_N); \tag{7}$$

hence, these two coefficients have equal absolute value. Since $D_N < 2u_{N+1}$ by (6), at least one of the degrees m and $D_N - m$ is below u_{N+1}. At that degree, the coefficient of P_N^{ε} agrees with the coefficient of any finite or infinite continuation of the product. Thus, for these exponent sequences, timidity of an infinite product implies timidity of all its finite partial products. In particular, this explains the finite consequence of Diao's even-level all-minus theorem mentioned in the Introduction.

We now turn to the cancellation needed for the direct proof. The next lemma determines the relative signs of two nonzero coefficients separated by the exponent of a factor.

Lemma 5 (Coefficient pair)**.** *Let d be a positive integer and $s \in \{-1, 1\}$. Suppose $f \in \mathbb{Z}[x]$ is a nonzero polynomial of degree less than $2d$, and both f and*

$$g = f(1 + sx^d)$$

are timid. If $g[m]$ and $g[m-d]$ are both nonzero, then

$$g[m]g[m-d] = s.$$

Proof. Set

$$a = f[m], \qquad b = f[m-d], \qquad c = f[m-2d].$$

The degree bound prevents a and c from both being nonzero, so $ac = 0$. Also,

$$g[m] = a + sb, \qquad g[m-d] = b + sc.$$

If $b = 0$, then

$$g[m]g[m-d] = sac = 0,$$

contrary to the hypothesis. Thus $b \in \{-1, 1\}$.

If $a \neq 0$, the sum $a + sb$ is either zero or has magnitude 2. This contradicts the fact that $g[m]$ is nonzero and g is timid. Hence $a = 0$. Likewise, $c \neq 0$ would make $b + sc$ either zero or of magnitude 2, so $c = 0$. It follows that

$$g[m]g[m-d] = (sb)b = s.$$

□

The recurrence for the exponents lets us apply this lemma to every coefficient overlap.

Lemma 6 (Reduction to an earlier product)**.** *Let $n > L$, put $q = n - L$, and define*

$$A_n = \prod_{i=q+1}^{n-1} \varepsilon_i.$$

Then, for every integer $t \geq 0$,

$$P_{n-1}[u_n + t] = A_n P_q[u_q + t]. \tag{8}$$

If also $0 \leq t \leq D_{q-1}$, then

$$P_{n-1}[t] = P_q[t]. \tag{9}$$

Both identities hold for arbitrary signs.

Proof. The exponent recurrence gives

$$D_{n-1} = u_n + D_{q-1}.$$

By (6), $D_{q-1} < u_{q+1}$. If a subset of the first $n-1$ indices omits any index $i \in \{q+1, \ldots, n-1\}$, its sum is at most

$$D_{n-1} - u_i \leq u_n + D_{q-1} - u_{q+1} < u_n.$$

Therefore every representation of $u_n + t$ using the first $n-1$ exponents must include all the parts $u_{q+1}, \ldots, u_{n-1}$.

Removing those parts subtracts

$$\sum_{i=q+1}^{n-1} u_i = u_n - u_q$$

from the degree and removes the weight A_n. This gives a bijection with representations of $u_q + t$ using the first q exponents, proving (8). The identity also holds when the relevant representation sets are empty.

Finally, if $0 \leq t \leq D_{q-1}$, then $t < u_{q+1}$. No exponent with index greater than q can occur in a representation of t, proving (9). □

Proposition 7 (Sufficiency)**.** *Every partial product associated with an admissible sign word or sequence is timid.*

Proof. Distinct subsets of the first L exponents have distinct sums. Indeed, at the largest index where two subsets differ, that exponent exceeds the sum of all preceding exponents. Thus every product P_n with $n \le L$ is timid, regardless of its signs.

By Lemma 3, every longer admissible word extends to an admissible infinite sequence. Fix such a sequence and proceed by induction on n. Let $n > L$, suppose all earlier products are timid, and put $q = n - L$.

By (4), only a degree at which both summands are nonzero could fail to have a timid coefficient. At such a degree we may write $m = u_n + t$, where

$$0 \le t \le D_{n-1} - u_n = D_{q-1}.$$

Notice that this interval contains all possible overlaps, but some of its degrees may have a zero summand.

Lemma 6 gives

$$P_{n-1}[m] = A_n P_q[u_q + t], \qquad P_{n-1}[t] = P_q[t].$$

At the overlap under consideration, both displayed coefficients of P_q are nonzero. Apply Lemma 5 with

$$f = P_{q-1}, \qquad g = P_q, \qquad d = u_q, \qquad s = \varepsilon_q.$$

The induction hypothesis gives timidity of f and g, and (6) gives

$$\deg P_{q-1} = D_{q-1} < 2u_q.$$

Consequently,

$$P_{n-1}[m]P_{n-1}[t] = A_n\varepsilon_q = \prod_{i=n-L}^{n-1} \varepsilon_i = -\varepsilon_n,$$

where the last equality is admissibility.

It follows that the two nonzero summands in

$$P_n[m] = P_{n-1}[m] + \varepsilon_n P_{n-1}[t]$$

have product -1 and therefore cancel. Every other coefficient has at most one nonzero summand and already belongs to $\{-1, 0, 1\}$. This completes the induction. □

3 The first failure

We now prove Theorem 2. The main step is to determine a periodic pattern of zero and nonzero coefficients in admissible products.

Fix an admissible infinite sequence η, and write

$$Q_n = P_n^{\eta}.$$

Let

$$T = \left\{ \sum_{i \in S} u_i : S \subsetneq \{1, \dots, L\} \right\}$$

be the set of sums over proper subsets of the initial block, and let

$$T_j = \left\{ \sum_{i \in S} u_i : S \subseteq \{1, \dots, j-1\} \right\} \qquad (1 \le j \le L).$$

The empty sum is included, so $T_1 = \{0\}$. Every T_j is a subset of T.

All subset sums of the initial block are distinct. Also, every later exponent is at least $u_{L+1} = D_L$. Therefore

$$\{t \in \mathbb{Z} : 0 \le t < u_{L+1},\ Q_n[t] \neq 0\} = T \qquad (n \ge L). \tag{10}$$

In particular,

$$Q_n[0] = 1, \qquad Q_n[u_j] = \eta_j \quad (1 \le j \le L,\ n \ge L). \tag{11}$$

The next lemma concerns only the coefficients at degrees $u_n + t$ with $t \in T$. It does not assert periodicity of the entire coefficient sequence.

Lemma 8 (Periodic pattern of shifted support)**.** *For $n \ge 1$, define*

$$E_n = \{t \in T : Q_n[u_n + t] \neq 0\}.$$

For $1 \le j \le L$ and every integer $h \ge 0$,

$$E_{j+2hL} = T_j, \qquad E_{j+(2h+1)L} = T \setminus T_j.$$

In particular, $E_{n+2L} = E_n$ for every $n \ge 1$. The least elements of T_j and $T \setminus T_j$ are 0 and u_j, respectively.

Proof. For $1 \le j \le L$, a subset of the first j exponents with sum at least u_j must contain u_j, because $D_{j-1} < u_j$. Deleting that part leaves a subset sum of $u_1, \dots, u_{j-1}$. All these sums occur with nonzero coefficient, so

$$E_j = T_j.$$

Now let $n > L$ and $t \in T$. Equation (10) shows that the second summand in

$$Q_n[u_n + t] = Q_{n-1}[u_n + t] + \eta_n Q_{n-1}[t]$$

is a nonzero sign. If the first summand is zero, the result is nonzero. If the first summand is nonzero, it too is a sign, and Proposition 7 forces the two summands to cancel. Thus

$$Q_n[u_n + t] \neq 0 \quad \Longleftrightarrow \quad Q_{n-1}[u_n + t] = 0.$$

By (8), applied to the products Q_n,

$$Q_{n-1}[u_n + t] \neq 0 \quad \Longleftrightarrow \quad Q_{n-L}[u_{n-L} + t] \neq 0.$$

It follows that

$$E_n = T \setminus E_{n-L}.$$

Together with $E_j = T_j$ for $1 \le j \le L$, this proves the formulas and the repetition after $2L$ indices.

Finally, 0 is the least element of T_j. By uniqueness of the initial subset sums, an element of $T \setminus T_j$ must use an initial exponent with index at least j, and hence is at least u_j. The singleton subset $\{j\}$ is proper because $L \ge 2$, so $u_j \in T$. Since $u_j > D_{j-1}$, it does not belong to T_j. Thus $\min(T \setminus T_j) = u_j$. □

Proof of Theorem 2. Let η be the admissible infinite sequence with the same first L signs as ε, and put $Q_n = P_n^{\eta}$. The recurrence (5) and the assumption that r is the first violating index imply

$$\varepsilon_i = \eta_i \quad (1 \le i < r), \qquad \varepsilon_r = -\eta_r.$$

In particular, $P_{r-1} = Q_{r-1}$.

Location and value. By (8),

$$Q_{r-1}[u_r + t] = \left(\prod_{i=r-L+1}^{r-1} \eta_i \right) Q_{r-L}[u_{r-L} + t] \qquad (t \geq 0).$$

Hence the least $t \in T$ for which $Q_{r-1}[u_r + t] \neq 0$ is $\min E_{r-L}$.

If $r \equiv j \pmod{2L}$ with $1 \leq j \leq L$, then Lemma 8 gives

$$E_{r-L} = T \setminus T_j, \qquad \min E_{r-L} = u_j.$$

If $r \equiv L + j \pmod{2L}$, then

$$E_{r-L} = T_j, \qquad \min E_{r-L} = 0.$$

Thus in both cases

$$\min E_{r-L} = d_r.$$

At $m = u_r + d_r$, the two summands in

$$Q_r[m] = Q_{r-1}[m] + \eta_r Q_{r-1}[d_r]$$

are nonzero signs. Indeed, the first is nonzero by the choice of d_r, and the second is nonzero because $d_r \in T$ and (10) applies. Timidity of Q_r forces cancellation. Reversing the last sign therefore gives

$$\begin{aligned} P_r[m] &= Q_{r-1}[m] - \eta_r Q_{r-1}[d_r] \\ &= -2\eta_r Q_{r-1}[d_r] \\ &= 2\varepsilon_r Q_{r-1}[d_r]. \end{aligned}$$

Equation (11), together with $\eta_j = \varepsilon_j$ for $j \leq L$, gives exactly the coefficient in the table.

Minimality and maximum magnitude at step r. Below degree u_r, the coefficients of P_r agree with those of the timid polynomial Q_{r-1}. It remains to consider degrees $u_r + t$ with $0 \leq t < d_r$. Here $t < u_{L+1}$.

If $t \notin T$, then $Q_{r-1}[t] = 0$ by (10), so only the unshifted summand contributes to $P_r[u_r + t]$. If $t \in T$, the choice of d_r gives $Q_{r-1}[u_r + t] = 0$, so only the shifted summand contributes. In either case, the coefficient has magnitude at most 1. This proves that $u_r + d_r$ is the least degree with a coefficient of magnitude greater than 1 in P_r.

Since P_r is the sum of two signed copies of the timid polynomial Q_{r-1}, every coefficient of P_r has magnitude at most 2. The coefficient already found attains that bound.

Persistence. If $d_r = 0$, then $u_r + d_r < u_{r+1}$ follows from strict increase of the exponents. If $d_r > 0$, its residue condition and $r \geq L + 1$ imply $r \geq 2L + 1$. The exponent recurrence then gives

$$u_{r+1} - u_r = \sum_{i=r-L+1}^{r-1} u_i \geq u_{r-1} > u_L \geq d_r.$$

Again $u_r + d_r < u_{r+1}$.

Every later factor has exponent at least u_{r+1}. It therefore changes none of the coefficients through degree $u_r + d_r$. Both the coefficient found above and the timidity of all lower coefficients persist under every continuation, finite or infinite. □

Proof of Theorem 1. Proposition 7 proves (iii) ⇒ (ii), and (ii) ⇒ (i) is immediate. If (iii) fails, let $r \le N$ be the first violating index. Theorem 2 supplies a coefficient of magnitude 2 in P_N. Thus (i) implies (iii).

The superincreasing initial block gives the assertion for $N < L$. The count of admissible words, their unique extensions, and the description of admissible infinite sequences follow from Lemma 3.

For an admissible infinite sequence, each coefficient of P_∞ agrees with that of a sufficiently long timid partial product. Thus P_∞ is timid. If the sequence is not admissible, Theorem 2 supplies a persistent coefficient of magnitude 2 in P_∞. This proves the infinite classification and its equivalence with timidity of every partial product. □

4 Examples and consequences

4.1 Fibonacci and tribonacci exponents

In the sign patterns below, + and − respectively denote +1 and −1. For a finite block B, the notation B^∞ denotes its infinite repetition.

Corollary 9 (Fibonacci sign patterns)**.** *Let $u_1 = a$, $u_2 = b$, where $0 < a < b$ are integers, and let*

$$u_{k+2} = u_{k+1} + u_k.$$

The infinite product

$$\prod_{k\ge1}(1 + \varepsilon_k x^{u_k})$$

is timid exactly for the following four sign sequences:

$$(-,-,-)^\infty, \qquad (-,+,+)^\infty, \qquad (+,-,+)^\infty, \qquad (+,+,-)^\infty.$$

For every $N \ge 1$, a finite product is timid if and only if its sign word is the length-N initial segment of one of these sequences. There are four distinct such words for every $N \ge 2$.

In particular, this classification applies to the Fibonacci exponents $1, 2, 3, 5, 8, \ldots$.

Proof. The hypotheses are precisely the level-2 conditions in (1). By Theorem 1, the admissible sequences repeat a block of three signs with product -1. Such a block has either three minus signs or exactly one minus sign, giving the four displayed possibilities. Their length-N prefixes exhaust the timid finite products; for $N = 1$, both possible signs occur. □

At level 2, the first-failure table takes the form

r mod 4	d_r	$P_N[u_r + d_r]$
0 or 3	0	$2\varepsilon_r$
1	a	$2\varepsilon_r\varepsilon_1$
2	b	$2\varepsilon_r\varepsilon_2$

for every $N \ge r$, including $N = \infty$ when a continuation is specified.

For the Fibonacci initialization $a = 1$, $b = 2$, the first coefficient of magnitude greater than 1 therefore occurs at one of the degrees u_r, $u_r + 1$, or $u_r + 2$. For example, the word

$$(-,-,-,-,+,\ldots)$$

first violates the rule at $r = 5$. Since $u_5 = 8$ and $d_5 = 1$, its least non-timid coefficient is

$$P_5[9] = -2,$$

and every continuation retains this coefficient.

Corollary 10 (Tribonacci sign patterns)**.** *Let u_1, u_2, u_3 be positive integers satisfying*

$$u_2 > u_1, \qquad u_3 > u_1 + u_2,$$

and let

$$u_{k+3} = u_{k+2} + u_{k+1} + u_k.$$

The infinite product

$$\prod_{k\geq 1}(1 + \varepsilon_k x^{u_k})$$

is timid exactly when the sign sequence repeats a block of four signs containing an odd number of minus signs. Explicitly, the eight possibilities are

$$(-,+,+,+)^\infty, \quad (+,-,+,+)^\infty, \quad (+,+,-,+)^\infty, \quad (+,+,+,-)^\infty,$$
$$(+,-,-,-)^\infty, \quad (-,+,-,-)^\infty, \quad (-,-,+,-)^\infty, \quad (-,-,-,+)^\infty.$$

For every $N \geq 1$, a finite product is timid if and only if its sign word is the length-N initial segment of one of these sequences. There are eight distinct such words for every $N \geq 3$.

In particular, the conclusion applies to the strictly increasing tribonacci normalization

$$u_1 = 1, \qquad u_2 = 2, \qquad u_3 = 4,$$

which gives

$$1, 2, 4, 7, 13, 24, 44, \ldots.$$

Proof. Apply Theorem 1 with $L = 3$. A block of four signs has product -1 exactly when it contains one or three minus signs. The displayed list gives all eight possibilities. Their prefixes also include every sign word of length at most 3, as required for the short finite products. □

For the initialization $(1, 2, 4)$, the offsets in Theorem 2 are

$r \bmod 6$	1	2	3	4	5	0
d_r	1	2	4	0	0	0

For example, the word

$$(-,+,+,+,-,+,-)$$

first violates the rule at $r = 7$. Here $u_7 = 44$ and $d_7 = 1$, so

$$P_7[45] = 2,$$

and this is its least non-timid coefficient. The coefficient persists under every continuation.

By contrast, the all-minus tribonacci word first violates the rule at $r = 4$, giving

$$P_N[7] = -2 \qquad (N \geq 4).$$

Thus the all-minus infinite product is excluded, but all eight patterns in Corollary 10 are included.

The initial inequalities matter. For example, the third-order recurrence starting with $(1, 2, 3)$ does not satisfy them. Even with the repeating sign block $(+,+,+,-)$, which satisfies the level-3 sign rule, the coefficient of x^3 is already 2, from the subsets $\{3\}$ and $\{1, 2\}$. The recurrence alone is therefore not sufficient for our classification.

4.2 Signed representation counts

For finite $N \geq 1$, let $R_N(m)$ be the number of subsets $S \subseteq \{1, \ldots, N\}$ with

$$\sum_{i \in S} u_i = m.$$

Let $R_\infty(m)$ count finite subsets of the full index set with the same sum. These counts are finite because the exponents are positive and tend to infinity.

Corollary 11 (Common support and coefficientwise minimality)**.** *Fix a finite $N \geq 1$ or $N = \infty$. Let η be an admissible sign word or sequence on the corresponding index set, and let ε be any sign choice on the same index set. For every $m \geq 0$,*

$$|P_N^\eta[m]| = \begin{cases} 1 & R_N(m) \text{ is odd,} \\ 0 & R_N(m) \text{ is even.} \end{cases}$$

In particular, all admissible products on that index set have the same support.

Moreover, for every $m \geq 0$,

$$|P_N^\eta[m]| \leq |P_N^\varepsilon[m]|.$$

Thus an admissible sign choice minimizes the absolute value of every coefficient simultaneously.

Proof. Modulo 2, every factor $1 + \varepsilon_k x^{u_k}$ equals $1 + x^{u_k}$. Consequently,

$$P_N^\varepsilon[m] \equiv R_N(m) \pmod 2$$

for every choice of signs.

By Theorem 1, P_N^η is timid. An integer in $\{-1, 0, 1\}$ is zero exactly when it is even, which proves the displayed formula and the common support. If $R_N(m)$ is odd, every possible coefficient at that degree is an odd integer and therefore has absolute value at least 1. If $R_N(m)$ is even, the claimed inequality is immediate because the admissible coefficient is zero. The same argument applies coefficientwise to the infinite product. □

At level 2, each of the three nonconstant admissible patterns places its minus signs in one residue class of indices modulo 3. For a fixed class, (2) counts representations using an even number of parts from that class minus representations using an odd number of parts from that class. Corollary 11 shows that all three differences vanish at exactly the same degrees as the all-minus difference, which instead records parity of the total number of parts.

Related enumeration problems. Klarner [9] studied distinct-part representation counts, including second-order recurrence sequences with arbitrary positive initial values. Further work on Fibonacci representation counts includes the papers of Weinstein [15], Chow and Slattery [4], and Sanna [12].

Stanley [14] studied coefficient power sums—i.e., sums $\sum_m c_m^p$, where the c_m are the coefficients of a finite product—for Fibonacci and higher-order exponent sequences. His products include factors $1 + tx^{u_k}$ with the same parameter t at every index. This common weighting differs from prescribing a separate sign ε_k for each available part, as in the classification here.

Chalebgwa and Szalay [3] studied representations of zero using generalized Fibonacci numbers with digits in $\{-1, 0, 1\}$, as well as related tribonacci representation counts. In that setting, a negative digit subtracts a term in the represented sum, and the digits vary between representations.

In (2), by contrast, every represented sum uses positive parts with multiplicities 0 or 1; the fixed sign ε_k affects only the weight assigned to a representation containing u_k.

Partitions in which a term may be used repeatedly form another distinct problem. Coons, Kristensen, and Laursen [5] studied asymptotic counts for such partitions over Fibonacci numbers and related sequences.

4.3 Superincreasing exponents and Pell-type recurrences

If the entire exponent sequence is superincreasing, rather than only its initial block, then distinct finite subsets have distinct sums. Indeed, at the largest index where two subsets differ, the corresponding exponent exceeds the sum of all preceding exponents. Every nonzero coefficient of a signed product therefore comes from exactly one subset and is a sign. Thus every sign choice produces timid partial products and a timid infinite product.

This gives a different classification from the quasifibonacci case, where the recurrence creates equal subset sums and forces cancellation conditions.

Proposition 12 (Pell-type exponents)**.** *Let $v_1 < v_2$ be positive integers and suppose*

$$v_{n+2} = cv_{n+1} + v_n,$$

where $c \geq 2$ is an integer. Every sign sequence produces timid partial products and a timid infinite product with exponents (v_k).

Proof. We prove by induction that

$$\sum_{i=1}^{n} v_i < v_{n+1} \qquad (n \geq 1).$$

The base case is $v_1 < v_2$. If the inequality holds at n, then

$$\sum_{i=1}^{n+1} v_i < 2v_{n+1} < cv_{n+1} + v_n = v_{n+2}.$$

Thus the entire sequence is superincreasing, and so the preceding unique-subset-sum argument applies. □

For more general increasing exponent sequences, the classification of timid sign choices depends on the equalities between their subset sums. The results here thus exhibit two contrasting cases: quasifibonacci exponents impose one cancellation rule on every consecutive block of $L + 1$ signs, whereas superincreasing exponents impose no restriction on the signs.

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